Higher order symmetries in a gauge covariant approach and quantum anomalies [*]

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Abstract

Higher order first integrals in a covariant Hamiltonian framework are investigated and the special role of the Killing-Yano tensors is pointed out. The covariant phase-space is extended to include external gauge fields and scalar potentials. Some nontrivial examples on a three-dimensional space involving Killing tensors of rank 2 are presented. It is shown that the conformal Killing vectors and tensors do not in general produce quantum operators that commute with the Klein-Gordon operator.

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1. Introduction

The concept of symmetries is one of the key concepts in physics, Noether's theorem giving a correspondence between symmetries and conservation laws.

The evolution of a dynamical system is described in the entire phase-space and from this point of view it is natural to go in search of conserved quantities to genuine symmetries of the complete phase-space, not just the configuration one. Such symmetries are associated to higher rank symmetric Stackel-Killing (SK) tensors which generalize the Killing vectors. These higher order symmetries are known as *hidden symmetries* and the corresponding conserved quantities are quadratic, or, more general, polynomial in momenta. Also Killing tensors play a pivotal role in Hamilton-Jacobi theory of separation of variables and the integrability of finite-dimensional

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Hamiltonian systems [1]. Another natural generalization of the Killing vectors is represented by the antisymmetric Killing-Yano (KY) tensors which in many respects are more fundamental than the KS tensors.

The conformal extension of the SK tensor equation determines the conformal Stackel-Killing (CSK) tensors which define first integrals of motion of the null geodesics. Investigations of the hidden symmetries of the higher dimensional space-times have pointed out the role of the conformal Killing-Yano (CKY) tensors to generate background metrics with black hole solutions (see, e. g. [2] for a brief review).

In the study of the dynamics of particles in external gauge fields it has been proved that a gauge covariant Hamiltonian approach of the symmetries [3] is more convenient and productive.

Passing from the classical motions to the hidden symmetries of a quantized system it is necessary to investigate the corresponding quantum constants and separability of the equations of motion. Especially in the case of hidden symmetries it can appear *anomalies* representing discrepancies between the conservation laws at the classical level and the corresponding ones at the quantum level.

The plan of the paper is as follows. In Section 2 we establish the generalized Killing equations in a covariant framework including external gauge fields and scalar potentials. In Section 3 we exemplify the gauge covariant approach with some nontrivial examples connected with the Kepler/Coulomb (KC) potential. In Section 4 we discuss the special role of the KY tensors in generating higher order symmetries. In the next Section we describe the relationship between conformal symmetries and the corresponding quantum operators in connection with quantum gravitational anomalies. Finally, the last Section is devoted to conclusions.

2. Symmetries and conserved quantities

Let us consider the Hamilton function describing the geodesic motion in a curved n-dimensional (pseudo-)Riemannian space $\mathcal{M}$ with the metric $\mathbf{g}$

$$H = \frac{1}{2}\, g^{ij} p_i p_j \,. \tag{1}$$

A conserved quantity of motion can be expanded as a power series in momenta:

$$K = K_0 + \sum_{k=1}^{p} \frac{1}{k!}\, K^{i_1\cdots i_k}(x) p_{i_1}\cdots p_{i_k}\,, \tag{2}$$

having a vanishing Poisson bracket with the Hamiltonian:

$$\{K, H\} = \frac{\partial K}{\partial x^i}\frac{\partial H}{\partial p_i} - \frac{\partial K}{\partial p_i}\frac{\partial H}{\partial x^i} = 0\,. \tag{3}$$

For the conservation of K its terms must satisfy

$$K^{(i_1\cdots i_k;i)} = 0\,, \tag{4}$$

where a semicolon denotes the covariant differentiation corresponding to the Levi-Civita connection ∇ and round brackets denotes full symmetrization over the indices enclosed. A symmetric tensor $K^{i_1\cdots i_k}$ satisfying (4) is called a SK tensor of rank k. Let us note that for any geodesic γ with tangent vector $\dot{x}^i = p^i$

$$Q_K = K_{i_1\cdots i_k}\dot{x}^{i_1}\cdots\dot{x}^{i_k}\,, \tag{5}$$

is constant along γ.

If a gauge field F_{ij} is present, the usual prescription is to replace the Hamiltonian (1) by

$$H = \frac{1}{2}\,g^{ij}(p_i - A_i)(p_j - A_j)\,, \tag{6}$$

where A_i are the potential 1-forms corresponding to the gauge field F_{ij}

$$F_{ij} = A_{j;i} - A_{i;j}\,. \tag{7}$$

For the conserved quantities of motion we consider the polynomials (2) in the variables $(p_i - A_i)$ and work out the Poisson bracket (3).

The disadvantage of this approach is that the canonical momenta p_i and implicitly the Hamilton equations of motion are not manifestly gauge covariant. This drawback can be removed using van Holten's receipt [3] by introducing the gauge invariant momenta:

$$\Pi_i = p_i - A_i\,. \tag{8}$$

The Hamiltonian (6) becomes

$$H = \frac{1}{2}\,g^{ij}\Pi_i\Pi_j + V(x)\,, \tag{9}$$

where, for completeness, a scalar potential $V(x)$ was included.

The equations of motion are derived using the Poisson bracket

$$\{P,Q\} = \frac{\partial P}{\partial x^i}\frac{\partial Q}{\partial \Pi_i} - \frac{\partial P}{\partial \Pi_i}\frac{\partial Q}{\partial x^i} + qF_{ij}\frac{\partial P}{\partial \Pi_i}\frac{\partial Q}{\partial \Pi_j}\,. \tag{10}$$

Now the fundamental Poisson brackets are

$$\{x^i, x^j\} = 0\,, \qquad \{x^i, \Pi_j\} = \delta^i_j\,, \qquad \{\Pi_i, \Pi_j\} = F_{ij}\,, \tag{11}$$

showing that the momenta Π_i are not canonical.

Searching for conserved quantities (2) expanded rather into powers of the gauge invariant momenta Π_i we get the following series of constraints

$$K^i V_{,i} = 0\,, \tag{12a}$$

$$K_0{}^{,i} + F_j{}^i K^j = K^{ij} V_{,j}\,. \tag{12b}$$

$$K^{(i_1\cdots i_l;i_{l+1})} + F_j{}^{(i_{l+1}} K^{i_1\cdots i_l)j} = \frac{1}{(l+1)} K^{i_1\cdots i_{l+1}j} V_{,j}\,, \quad for \;\; l = 1\,, \cdots (p-2)\,, \tag{12c}$$

$$K^{(i_1\cdots i_{p-1};i_p)} + F_j{}^{(i_p} K^{i_1\cdots i_{p-1})j} = 0\,, \tag{12d}$$

$$K^{(i_1\cdots i_p;i_{p+1})} = 0\,. \tag{12e}$$

Examining the above hierarchy of constraints some remarks are in order. In the presence of the gauge field F_{ij} only the last equation (12e) corresponds to a SK tensor (4), while the rest of the equations mixes up the terms of K with the gauge field strength and potential $V(x)$. Also it is worth mentioning that equations (12) separate into two groups: one involves the terms of K of odd degree in the momenta Π_i and the other involves only terms of K of even degree in the momenta [4].

Several applications using van Holten's covariant framework [3] are given in [5, 6, 7, 8]. In what follows we shall exemplify the gauge covariant approach with a few simple but not trivial examples.

3. Explicit examples

Let us illustrate these general considerations by some nontrivial examples. In what follows we consider $\mathcal{M}$ to be a 3-dimensional Euclidean space $\mathbb{E}^3$ and in these circumstances it is more convenient to get rid of a difference between covariant and contravariant indices. The KC potential will be the basis of our examples adding different other electromagnetic field. The hidden symmetries which will be found involve SK tensors of rank 2.

3.1. Coulomb potential

We investigate the constants of motion in a KC potential superposing different types of electric and magnetic fields. To put in a concrete form, we consider the Hamiltonian for the motion of a point charge q of mass M in the Coulomb potential produced by a charge Q

$$H = \frac{M}{2}\dot{\mathbf{r}}^2 + q\frac{Q}{r}\,. \tag{13}$$

The most general SK tensor of rank 2 in a 3-dimensional Euclidean space turns out to be [9]:

$$K_{ij} = \epsilon_{km(i}\epsilon_{j)ln} A_{mn} x_k x_l + (B_{l(i}\epsilon_{j)kl} + \lambda_{(i}\delta_{j)k} - \delta_{ij}\lambda_k) x_k + C_{ij}\,, \tag{14}$$

where A_{mn}, B_{li} and C_{ij} are constant tensors. However, for the KC problem it proved that the following form is adequate [9]:

$$K_{ij} = 2\delta_{ij}\mathbf{n}\cdot\mathbf{r} - (n_i r_j + n_j n_i)\,, \tag{15}$$

written in spherical coordinates with $\mathbf{n}$ an arbitrary constant vector.

Corresponding to this SK tensor the non relativistic KC problem admits the Runge-Lenz vector constant of motion

$$\mathbf{K} = \mathbf{\Pi}\times\mathbf{L} + MqQ\frac{\mathbf{r}}{r}\,, \tag{16}$$

where

$$\mathbf{L} = \mathbf{r}\times\mathbf{\Pi}\,, \tag{17}$$

is the angular momentum.

3.2. Constant electric field

The next more involved example consists of an electric charge q moving in the Coulomb potential in the presence of a constant electric field $\mathbf{E}$. The corresponding Hamiltonian is:

$$H = \frac{1}{2M}\,\mathbf{\Pi}^2 + q\frac{Q}{r} - q\mathbf{E}\cdot\mathbf{r}\,, \tag{18}$$

with $\mathbf{\Pi} = M\dot{\mathbf{r}}$ in spherical coordinates of $\mathbb{E}^3$.

We are looking for a constant of motion for the system governed by the Hamiltonian (18) of the form

$$K = K_0 + K_i\Pi_i + \frac{1}{2}\,K_{ij}\Pi_i\Pi_j\,. \tag{19}$$

As we observed in Section 2, the last equation of the system (12) for $p = 2$ is satisfied by a SK tensor of rank 2. Again it is adequate to choose for the SK tensor of rank 2 the simple form (15).

In the presence of a constant electric field $\mathbf{E}$ it proves convenient to choose $\mathbf{n}$ along $\mathbf{E}$ and we start to solve the hierarchy of constraint (12) with a solution of equation (12e) of the form (15) with $\mathbf{n} = \mathbf{E}$. Using this form for K_{ij} and the derivative of the potential V corresponding to the Hamiltonian (18)

$$V_{,i} = -\frac{qQ}{r^3}r_i - qE_i\,, \tag{20}$$

we get from (12b) after a straightforward calculation

$$K_0 = \frac{MqQ}{r}\mathbf{E}\cdot\mathbf{r} - \frac{Mq}{2}\,\mathbf{E}\cdot[\mathbf{r}\times(\mathbf{r}\times\mathbf{E})]\,. \tag{21}$$

Concerning equation (12a) in conjunction with (20), it is automatically satisfied with a vector **K** of the form

$$\mathbf{K} = \mathbf{r} \times \mathbf{E}\,, \tag{22}$$

modulo an arbitrary constant factor. This vector **K** contribute to a conserved quantity with a term proportional to the angular momentum **L** along the direction of the electric field **E**.

In conclusion, when a uniform constant electric field is present, the KC system admits two constants of motion $\mathbf{L} \cdot \mathbf{E}$ and $\mathbf{C} \cdot \mathbf{E}$ where **C** is a generalization of the Runge-Lenz vector (16):

$$\mathbf{C} = \mathbf{K} - \frac{Mq}{2}\mathbf{r} \times (\mathbf{r} \times \mathbf{E})\,. \tag{23}$$

3.3. Spherically symmetric magnetic field

Another configuration which admits a hidden symmetry is the superposition of an external spherically symmetric magnetic field

$$\mathbf{B} = f(r)\mathbf{r}\,, \tag{24}$$

over the Coulomb potential acting on a electric charge q. This configuration is quite similar to the Dirac charge-monopole system.

For the beginning the scalar function $f(r)$ is not fixed, its form will be determined from the hierarchy of constraints (12). For K_{ij} we use again the form (15) and F_{ij} in this case is

$$F_{ij} = \epsilon_{ijk} B_k = \epsilon_{ijk} r_k f(r)\,. \tag{25}$$

From (12d) for $l = 1$ we have

$$K_{(i,j)} = -qf(r)[(\mathbf{n} \times \mathbf{r})_{(i} r_{j)}]\,, \tag{26}$$

with **n** an arbitrary unit constant vector. It is easy to get for the vector **K** the solution

$$K_i = q\left[\int rf(r)dr\right](\mathbf{n} \times \mathbf{r})_i\,, \tag{27}$$

and equation (12a) is obviously satisfied.

For K_0, equation (12b) can be solely solved making choice of a definite form for the function $f(r)$

$$f(r) = \frac{g}{r^{5/2}}\,, \tag{28}$$

with g a constant connected with the strength of the magnetic field. For this function $f(r)$ the energy of the magnetic field diverges at $r = 0$ and $r \to \infty$. Of course such a special magnetic field (24) could be prepared only in a finite region of space and all present considerations are limited to this space domain.

With this special form of the function $f(r)$ we get

$$K_0 = \left[\frac{MqQ}{r} - \frac{2g^2q^2}{r}\right](\mathbf{n}\cdot\mathbf{r})\,, \tag{29}$$

and

$$K_i = -\frac{2gq}{r^{1/2}}(\mathbf{r}\times\mathbf{n})_i\,. \tag{30}$$

Collecting the terms K_0, K_i, K_{ij} the constant of motion (19) becomes

$$K = \mathbf{n}\cdot\left(\mathbf{K} + \frac{2gq}{r^{1/2}}\,\mathbf{L} - 2g^2q^2\frac{\mathbf{r}}{r}\right)\,, \tag{31}$$

with $\mathbf{n}$ an arbitrary constant unit vector and $\mathbf{K}, \mathbf{L}$ given by (16), (17) respectively. The angular momentum $\mathbf{L}$ [3] is not separately conserved, entering the constant of motion (31).

3.4. Magnetic field along a fixed direction

The last example consists in a magnetic field directed along a fixed unit vector $\mathbf{n}$

$$\mathbf{B} = B(\mathbf{r}\cdot\mathbf{n})\mathbf{n}\,, \tag{32}$$

where, for the beginning, $B(\mathbf{r}\cdot\mathbf{n})$ is an arbitrary function.

Again we are looking for a constant of motion of the form (19) with the SK tensor of rank 2 (15). Equation (12c) for $l = 1$ reads

$$K_{(i,j)} = -qB(\mathbf{r}\times\mathbf{n})_{(i}n_{j)}\,. \tag{33}$$

K_i, solution of this equation, must satisfy also equation (12a) and after some straightforward calculations we get

$$K_i = q\left[\int rB(\mathbf{r}\cdot\mathbf{n})d(\mathbf{r}\cdot\mathbf{n})\right](\mathbf{r}\times\mathbf{n})_i\,. \tag{34}$$

Equation (12b) for K_0 proves to be solvable only for a particular form of the magnetic field

$$\mathbf{B} = \frac{\alpha}{\sqrt{\alpha\mathbf{r}\cdot\mathbf{n}+\beta}}\,\mathbf{n}\,, \tag{35}$$

with α and β two arbitrary constants.

Finally we get for K_0 and K_{1i}

$$K_0 = \frac{MqQ}{r}(\mathbf{r}\cdot\mathbf{n}) + \alpha q^2(\mathbf{r}\times\mathbf{n})^2\,, \tag{36}$$

$$K_i = -2q\sqrt{\alpha\mathbf{r}\cdot\mathbf{n}+\beta}\,(\mathbf{r}\times\mathbf{n})_i\,. \tag{37}$$

The final form of the conserved quantity in this case is:

$$K = \mathbf{n}\cdot\left[\mathbf{K} + 2q\sqrt{\alpha\mathbf{r}\cdot\mathbf{n}+\beta}\ \ \mathbf{L}\right] + \alpha q^2(\mathbf{r}\times\mathbf{n})^2\,. \tag{38}$$

As in the previous example the angular momentum $\mathbf{L}$ is forming part of the constant of motion K (38).

4. Killing-Yano tensors

Killing-Yano (KY) tensors are a different generalization of Killing vectors which can be studied on a manifold. They were introduced by Yano [10] from a purely mathematical perspective and later on it turned out they have many interesting properties relevant to physics. For the first time Floyd [11] and Penrose [12] showed that the SK tensor of the four-dimensional space-time admits a square root in terms of KY tensors. After that Carter and McLenaghan [13] were able to construct from KY tensors Dirac type operators which commute with the standard Dirac operator. Gibbons *et al* [14] stressed the role of the KY tensors of rank 2 in general relativity and in dynamics of spin $\frac{1}{2}$ fermions in spinning particle models. In recent years the KY tensors are related to a multitude of different topics such supersymmetries, index theorems, supergravity theories, and so on [15].

A KY tensor $Y_{i_1\cdots i_p}$ is antisymmetric satisfying the following equation:

$$Y_{i_1\cdots i_{p-1}(i_p;j)} = 0\,. \tag{39}$$

The first connection with the symmetry properties of the geodesic motion is the observation that along every geodesic γ in $\mathcal{M}$, $Y_{i_1\cdots i_{p-1}j}\dot{x}^j$ is parallel.

These two generalizations SK and KY of the Killing vectors could be related. Let $Y_{i_1\cdots i_p}$ be a KY tensor, then the symmetric tensor field

$$K_{ij} = Y_{ii_2\cdots i_p}Y_j^{\ i_2\cdots i_p}\,, \tag{40}$$

is a SK tensor and it sometimes refers to this SK tensor as the associated tensor with $Y_{i_1\cdots i_p}$. That is the case of the Kerr metric [11, 12] or the Euclidean Taub-NUT space [16, 17]. However, the converse statement is not true in general: not all SK tensors of rank 2 are associated with a KY tensor. A counterexample is the generalized Taub-NUT space [18] which admit SK tensors but no KY tensors [19].

Having in mind the special role of null geodesic for the motion of massless particles, it is convenient to look for conformal generalization of KY tensor. Let us mention also that recently a lot of interest focuses on higher dimensional black holes. It was demonstrated the remarkable role of the conformal Killing-Yano (CKY) tensors in the study of the properties of such black holes (see e. g. [20, 21, 22] and the cites contained therein). In what follows we limit ourselves to CKY tensors of rank 2 which satisfy [23]

$$Y_{ij;k} + Y_{kj;i} = \frac{2}{n-1}\left(g_{ki}Y^l_{\ j;l} + g_{j(i}Y_{k)\ \ ;l}^{\ \ l}\right)\,. \tag{41}$$

There is also a conformal generalization of the SK tensors, namely a symmetric tensor $K_{i_1\cdots i_p} = K_{(i_1\cdots i_p)}$ is called a conformal Killing (CSK) tensor if it obeys the equation

$$K_{(i_1\cdots i_p;j)} = g_{j(i_1}\tilde{K}_{i_2\cdots i_p)}\,, \tag{42}$$

where the tensor $\tilde{K}$ is determined by tracing the both sides of equation (42). Let us note that in the case of CSK tensors, the quantity (5) is constant only for null geodesics γ. There is also a similar relation between CKY and CSK tensors as in equation (40). Namely if Y_{ij} is a CKY tensor

$$K_{ij} = Y_i{}^k Y_{kj}\,, \tag{43}$$

is a CSK tensor [23].

5. Quantum gravitational anomalies

In order to find the necessary conditions for the existence of constants of motion in a first-quantize system we replace momenta by derivatives and look for operators commuting with the Hamiltonian:

$$\mathcal{H} = \Box = \nabla_i g^{ij}\nabla_j = \nabla_i\nabla^i\,, \tag{44}$$

corresponding to a free scalar particle and the covariant Laplacian is acting on scalars.

Many times the classical conserved quantities associated with SK tensors do not generally transfer to the quantized systems producing so-called quantum anomalies [24]. In what follows we shall analyze the quantum anomalies in the case of SK and CSK tensors and, to make things more specific, we confine ourselves to the case of tensors of rank 1 and 2.

For the beginning let us consider the conserved operator corresponding to a conformal Killing vector K^i in the quantized system

$$\mathcal{Q}_V = K^i\nabla_i\,. \tag{45}$$

In order to identify a quantum gravitational anomaly we shall evaluate the commutator $[\Box, \mathcal{Q}_V]\Phi$ for $\Phi \in \mathcal{C}^\infty(\mathcal{M})$, solutions of the Klein-Gordon equation with the Klein-Gordon operator (44). A straightforward calculation gives

$$[\mathcal{H}, \mathcal{Q}_V] = \frac{2-n}{n}K_k{}^{;ki}\nabla_i + \frac{2}{n}K^k{}_{;k}\Box\,. \tag{46}$$

As it is expected, in the case of ordinary Killing vectors the r. h. s. of this commutator vanishes and there are no quantum gravitational anomalies. But for conformal Killing vectors, we confront with a quite different situation. Even if we evaluate the r. h. s. of (46) on solutions of the massless

Klein-Gordon equation, $\Box\Phi(x) = 0$, the term ${K_k}^{;ki}\nabla_i$ survives. It is possible to find some particular spaces in which the r. h. s. of (46) vanishes, but in general the system is affected by quantum gravitational anomalies.

In the case of SK and CSK tensors of rank 2 the quantum analog of the classical conserved quantity (2) is

$$\mathcal{Q}_T = \nabla_i K^{ij} \nabla_j \,, \tag{47}$$

where K^{ij} is a SK (4) or a CSK (42) tensor.

Now the evaluation of the commutator between $\mathcal{H}$ and the quantum operator (47) is more involved than in case of Killing vectors and the result of this tedious evaluation is [25, 26]:

$$\begin{aligned}
[\Box, \mathcal{Q}_T] = {} & 2\left(\nabla^{(k}K^{ij)}\right)\nabla_k\nabla_i\nabla_j \\
& +3\nabla_m\left(\nabla^{(k}K^{mj)}\right)\nabla_j\nabla_k \\
& +\left\{-\frac{4}{3}\nabla_k\left(R_m^{[k}K^{j]m}\right)\right. \\
& \left.+\nabla_k\left(\frac{1}{2}g_{ml}(\nabla^k\nabla^{(m}K^{lj)} - \nabla^j\nabla^{(m}K^{kl)}) + \nabla_i\nabla^{(k}K^{ij)}\right)\right\}\nabla_j \,.
\end{aligned} \tag{48}$$

We mention that the last term is missing in the corresponding equation in [24]. Note also that the terms are arranged into groups with three, two and just one derivatives and consequently it is impossible to have compensations between them.

However in the case of SK tensors all the symmetrized derivatives vanish and we end up with a simpler result

$$[\Box, \mathcal{Q}_T] = -\frac{4}{3}\nabla_k({R_m}^{[k}K^{j]m})\nabla_j \,. \tag{49}$$

Since the r. h. s. does not vanish identically, SK tensors exhibit quantum anomalies, i. e. the classical conservation law does not transfer to the quantum level.

Although in general SK tensors do not give quantum mechanical symmetries, there are a few notable conditions for which the r. h. s. of (49) does vanish. The simplest case in which this happens is obviously when the space is Ricci flat. A slightly more general case is when the space is Einstein, i. e. $R_{ij} \propto g_{ij}$ (that is, if the vacuum Einstein equations are fulfilled). In this case we get that the r. h. s. of the commutator involves $K^{[ij]}$ and consequently vanishes since SK tensors are symmetric. A more interesting and quite unexpected case is represented by SK tensors associated to KY tensors of rank 2 as in (40) [24] a situation which occurs for some spaces [11, 12, 16, 17].

For CSK tensors the existence of some favorable conditions which could prevent the appearance of quantum anomalies is quite impossible. In the case of CSK tensors we could not simplify any more the commutator (48) since for them the symmetrized derivatives do not vanish. Even if we evaluate the commutator for CSK tensors associated with CKY tensors we do not obtain a cancellation of anomalies [26]. Therefore we are not able to find any favorable circumstances on the CSK tensors in order to achieve a conserved quantum operator.

6. Concluding comments

The (C)SK and (C)KY tensors are related to a multitude of different topics such as classical integrability of systems together with their quantization, supergravity, string theories, hidden symmetries in higher dimensional black-holes space-times, etc.

To conclude let us discuss shortly some problems that deserve a further attention. An obvious extension of the gauge covariant approach to hidden symmetries is represented by the non-Abelian dynamics using the appropriate Poisson brackets [3, 5]. We worked out some examples in an Euclidean 3-dimensional space and restricted to SK tensors of rank 2. More elaborate examples working in a N-dimensional curved space and involving higher ranks of SK tensors [27] will be presented elsewhere [28].

The concept of generalized (C)KY symmetry in the presence of a skew-symmetric torsion is more widely applicable and may become very powerful [29].

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