Periodic and stationary wave solutions of two component Zakharov-Yajima-Oikawa system, using Madelung approach *

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Abstract

To describe the propagation of three nonlinear pulses in some dispersive material one have to solve simultaneously a set of coupled NLS equations. Using a multiple scales method the interaction between two bright and one dark solitons is studied and an Yajima-Oikawa completely integrable system is obtained. The one-soliton solutions of the corresponding Yajima-Oikawa system are determined using a Madelung fluid description.

PACS: 42.81 Dp, 05.45.Yv
MSC2000: 35Q55, 37K10, 45G15

Basic equations

Extending the work of Kivshar [1] for a bright-dark soliton interaction, one considers a multimode optical fiber with three nonlinear dispersive waves in interaction. Suppose that the dispersion relations of these weakly nonlinear waves are $\omega_i = \omega_i(k_i : |A_1|^2, |A_2|^2, |A_3|^2), \quad i = 1, 2, 3$
Let $e^{i(k_0 x - \omega_0 t)}$ be a carrier wave.
A Taylor expansion around (k_0, ω_0) and $|A_i| = 0$ of each ω_i gives

$$\omega_i - \omega_0 = \left(\frac{\partial \omega_i}{\partial k_i}\right)_0 (k_i - k_0) + \frac{1}{2}\left(\frac{\partial^2 \omega_i}{\partial k_i^2}\right)_0 (k_i - k_0)^2 + \left(\frac{\partial \omega_i}{\partial |A_1|^2}\right)_0 |A_1|^2 + \left(\frac{\partial \omega_i}{\partial |A_2|^2}\right)_0 |A_2|^2 + \left(\frac{\partial \omega_i}{\partial |A_3|^2}\right)_0 |A_3|^2 + ... \qquad (1)$$

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Replacing $\omega_i - \omega_0 \simeq -i\frac{\partial}{\partial t}$, $\quad k_i - k_0 \simeq i\frac{\partial}{\partial x}$, after a translation of coordinate $(x \to x - \left(\frac{\partial \omega_3}{\partial k_3}\right)_0 t)$, we obtain a nonlinear system of three interacting waves

$$\begin{aligned}
&i\frac{\partial A_1}{\partial t} + iV_1\frac{\partial A_1}{\partial x} + \frac{\alpha_1}{2}\frac{\partial^2 A_1}{\partial x^2} + \alpha_2|A_1|^2 A_1 + \alpha_3|A_2|^2 A_1 + \alpha_4|A_3|^2 A_1 = 0\\
&i\frac{\partial A_2}{\partial t} + iV_2\frac{\partial A_2}{\partial x} + \frac{\beta_1}{2}\frac{\partial^2 A_2}{\partial x^2} + \beta_2|A_1|^2 A_2 + \beta_3|A_2|^2 A_2 + \beta_4|A_3|^2 A_2 = 0\\
&i\frac{\partial A_3}{\partial t} + \frac{\gamma_1}{2}\frac{\partial^2 A_3}{\partial x^2} + \gamma_2|A_1|^2 A_3 + \gamma_3|A_2|^2 A_3 + \gamma_4|A_3|^2 A_3 = 0,
\end{aligned} \tag{2}$$

where $V_i = \left(\frac{\partial \omega_i}{\partial k_i}\right)_0 - \left(\frac{\partial \omega_3}{\partial k_3}\right)_0$, $\quad i = 1,2$ and the constants $\alpha_1, \beta_1, \gamma_1$ are related to derivatives of ω_i with respect to k_i $(\alpha_1 = -\left(\frac{\partial^2 \omega_1}{\partial k_1^2}\right)_0, ...)$ and $\alpha_2, ...\gamma_4$ to the derivatives with respect to $|A_i|^2$ $\quad \left(\alpha_2 = \left(\frac{\partial \omega_1}{\partial |A_1|^2}\right), ...\right)$.

Further on we consider channel 3 normal- and 1 and 2 with anomalous dispersion [1], [2]. Writing

$$A_1 = \Psi_1 e^{i\delta_1 t}, \quad A_2 = \Psi_2 e^{i\delta_2 t}, \quad A_3 = (u_0 + a(x,t))e^{i(\Gamma t + \phi(x,t))}$$

$$\delta_i = -\left(\frac{\partial \omega_i}{\partial |A_3|^2}\right)_0 u_0^2, \qquad \Gamma = -\left(\frac{\partial \omega_3}{\partial |A_3|^2}\right)_0 u_0^2.$$

$(u_0,\ a$ real) the system (2) transforms into

$$\begin{aligned}
&i\frac{\partial \Psi_1}{\partial t} + iV_1\frac{\partial \Psi_1}{\partial x} + \frac{\alpha_1}{2}\frac{\partial^2 \Psi_1}{\partial x^2} + \left(\alpha_2|\Psi_1|^2 + \alpha_3|\Psi_2|^2\right)\Psi_1 + 2\alpha_4 u_0 a \Psi_1 = 0\\
&i\frac{\partial \Psi_2}{\partial t} + iV_2\frac{\partial \Psi_2}{\partial x} + \frac{\beta_1}{2}\frac{\partial^2 \Psi_2}{\partial x^2} + \left(\beta_2|\Psi_1|^2 + \beta_3|\Psi_2|^2\right)\Psi_2 + 2\alpha_4 u_0 a \Psi_2 = 0 \quad (3)\\
&\frac{\partial^2 a}{\partial t^2} + \gamma_1\gamma_2 u_0^2\frac{\partial^2 a}{\partial x^2} + \frac{\gamma_1^2}{4}\frac{\partial^4 a}{\partial x^4} + \frac{\gamma_1}{2}\frac{\partial^2}{\partial x^2}(\gamma_2 u_0|\Psi_1|^2 + \gamma_3 u_0|\Psi_2|^2)\\
&\qquad + \textit{nonlinear terms in } (a,\phi) = 0
\end{aligned}$$

The linear part of the a equation corresponds to an acoustic field with dispersion relation $(\gamma_1 < 0,\ \gamma_4 > 0)$

$$\omega = ck\sqrt{1 + \frac{\gamma_1^2}{4c^2}k^2} \simeq ck\left(1 + \frac{\gamma_1^2}{8c^2}k^2\right)$$

and phase velocity $c = \omega/k$, where $c^2 = |\gamma_1|\gamma_4 u_0^2$.

We shall perform a multiple scales analysis of the system (3). We introduce new functions and scaled variables by

$$t \Rightarrow \epsilon t, \quad x \Rightarrow \sqrt{\epsilon}(x - ct), \quad a \Rightarrow \epsilon a, \quad \phi \Rightarrow \epsilon\phi, \quad \Psi_1 \Rightarrow \epsilon^{\frac{3}{4}}\Psi_1, \quad \Psi_2 \Rightarrow \epsilon^{\frac{3}{4}}\Psi_2,$$

In order $\frac{5}{4}$ in ϵ from a equation we obtain

$$c\frac{\partial a}{\partial t} + \frac{\gamma_1}{2}\frac{\partial}{\partial x}\left(\gamma_2 u_0|\Psi_1|^2 + \gamma_3 u_0|\Psi_2|^2\right) = 0. \tag{4}$$

All the nonlinear terms in a equation contribute to higher order in ϵ. In order $\frac{5}{4}$ from Ψ_i equations we obtain $V_1 = V_2 = c$, which represents a long wave-short wave resonance condition [3].

In the next order $\left(\frac{7}{4}\right)$ in ϵ from the Ψ equations we get

$$\begin{aligned} i\frac{\partial \Psi_1}{\partial t} + \frac{\alpha_1}{2}\frac{\partial^2 \Psi_1}{\partial x^2} + 2\alpha_4 u_0 a \Psi_1 = 0 \\ i\frac{\partial \Psi_2}{\partial t} + \frac{\beta_1}{2}\frac{\partial^2 \Psi_2}{\partial x^2} + 2\beta_4 u_0 a \Psi_2 = 0 \end{aligned} \tag{5}$$

The equations (4), (5) represent an 1D 2 component Zakharov [4], Yajima-Oikawa [5] system. A similar problem in 2-D was considered in [6].

Madelung's Approach

The special case of coefficients $(\alpha_i = \beta_i,\ \ \gamma_2 = \gamma_3)$ is integrable [6] and will be considered in what follows. In this case, simplifying the notations we have to study the system

$$\begin{aligned} &\frac{\partial a}{\partial t} + \gamma\frac{\partial}{\partial x}\left(|\Psi_1|^2 + |\Psi_2|^2\right) = 0 \\ &i\frac{\partial \Psi_i}{\partial t} + \frac{1}{2}\frac{\partial^2 \Psi_i}{\partial x^2} + \beta\Psi_i a = 0, \qquad i = 1, 2. \end{aligned} \tag{6}$$

Following Madelung [7] we consider

$$\Psi_i = \sqrt{\rho_i}\ e^{i\theta_i}.$$

Then from the first equation (6) we have

$$\frac{\partial a}{\partial t} + \gamma\frac{\partial}{\partial x}(\rho_1 + \rho_2) = 0 \tag{7}$$

and from the Ψ_i equations we obtain the continuity equations for the fluid densities ρ_i

$$\begin{aligned} &\frac{\partial \rho_i}{\partial t} + \frac{\partial}{\partial x}(v_i\rho_i) = 0, \qquad i = 1, 2 \\ &v_i(x,t) = \frac{\partial \theta_i(x,t)}{\partial x}, \end{aligned} \tag{8}$$

and

$$-\frac{\partial\theta_i}{\partial t} + \frac{1}{2}\frac{1}{\sqrt{\rho_i}}\frac{\partial^2\sqrt{\rho_i}}{\partial x^2} - \frac{1}{2}\left(\frac{\partial\theta_i}{\partial x}\right)^2 + \beta a = 0. \tag{9}$$

Derivating this last ones with respect to x the following equations of motion for the fluid velocities v_i are obtained

$$\left(\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x}\right)v_i = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho_i}}\frac{\partial^2\sqrt{\rho_i}}{\partial x^2}\right) + \beta\frac{\partial a}{\partial x}. \tag{10}$$

Following Fedele [8] the equations (10) can be written as

$$-\rho_i\frac{\partial v_i}{\partial t} + v_i\frac{\partial\rho_i}{\partial t} + 2\left[c_i(t) - \int\frac{\partial v_i}{\partial t}\,dx\right]\frac{\partial\rho_i}{\partial x}$$

$$+\frac{1}{4}\frac{\partial^3\rho_i}{\partial x^3} + \rho_i\frac{\partial}{\partial x}(a\rho_i) + 2a\rho_i\frac{\partial\rho_i}{\partial x} = 0 \tag{11}$$

with c_i arbitrary integration quantities, eventually dependent on time.

Motion with constant velocity ($v_1 = v_2 = v_0$

From the equations of continuity one can see that both $\rho_1(x,t)$ and $\rho_2(x,t)$ depend on $\xi = x - v_0 t$. We assume that also $a(x,t)$ depends only on ξ. Denoting $-E_i = 2c_i - v_0^2$ the equations of motion satisfied by ρ_i write

$$\frac{1}{4}\frac{d^3\rho_i}{d\xi^3} - E_i\frac{d\rho_i}{d\xi} + 2a\frac{d\rho_i}{d\xi} + \rho_i\frac{da}{d\xi} = 0. \tag{12}$$

From the a equation it is easily seen that

$$a = \mu(\rho_1 + \rho_2), \qquad \frac{\gamma}{v_0} = \mu. \tag{13}$$

We shall discuss firstly the situation $E_1 = E_2$. Then the equations (12) write

$$\frac{1}{4}\frac{d^3\rho_i}{d\xi^3} - E\frac{d\rho_i}{d\xi} + \mu\rho_i\frac{d}{d\xi}(\rho_1+\rho_2) + 2\mu(\rho_1+\rho_2)\frac{d\rho_i}{d\xi} = 0 \tag{14}$$

the same [12], [10] as for the Manakov system [9]. Solutions for Manakov model were studied by many authors ([12], where many other references can be found). In what follows we shall present some solutions of (13) which can be obtained using Madelung approach (for more details see [11], [10].

It is convenient to add the two equations of motion (14); denoting $z_+ = \rho_1 + \rho_2 \quad (\xi \to 2\xi)$ we get

$$\frac{d^3 z_+}{d\xi^3} - E\frac{dz_+}{d\xi} + \frac{3}{2}\mu\frac{d}{d\xi}z_+^2 = 0 \tag{15}$$

Integrating twice one obtains

$$\frac{1}{4}\left(\frac{dz_+}{d\xi}\right)^2 = -\mu z_+^3 + E z_+^2 + A z_+ + B = P_3(z_+) \tag{16}$$

Subtracting the two equations of motion (14) and denoting $z_- = \rho_1 - \rho_2$ we find

$$\frac{1}{4}\frac{d^3 z_-}{d\xi^3} - E\frac{dz_-}{d\xi} + \mu z_- \frac{dz_+}{d\xi} + 2\mu z_+ \frac{dz_-}{d\xi} = 0 \tag{17}$$

a linear differential equation in z_- (once $z_+(\xi)$ is known) A special solution is

$$z_- = (p_1^2 - p_2^2) z_+ \qquad\qquad p_1^2 + p_2^2 = 1 \tag{18}$$

Then

$$\rho_1 = p_1^2 z_+, \qquad \rho_2 = p_2^2 z_+ \tag{19}$$

For constant velocities the densities ρ_i have to satisfy additional conditions, namely

$$\frac{1}{2}\frac{1}{\sqrt{\rho_i}}\frac{\partial^2 \sqrt{\rho_i}}{\partial x^2} + \mu z_+(\xi) = \lambda_i, \tag{20}$$

which for the previous solutions (19) becomes

$$\frac{d^2 z_+}{d\xi^2} - \frac{1}{2z_+}\left(\frac{dz_+}{d\xi}\right)^2 + \mu z_+ - \lambda z_+ = 0. \tag{21}$$

It can be satisfied if $\lambda = \frac{E}{2}$ $\qquad B = 0.$
Assume $P_3(z_+)$ has three distinct roots; then

$$P_3(z_+) = -\mu(z_+ - z_1)(z_+ - z_2)(z_+ - z_3) \tag{22}$$

The restriction $B = 0$ means that one of the roots z_2 or z_3 is 0. We obtained two acceptable periodic solutions

$$z_1 > 0, \qquad z_2 = 0, \qquad z_3 < 0$$

$$z_+ = z_1 \ \mathsf{cn}^2 u \tag{23}$$

$$u = \frac{2\sqrt{\mu}}{g}\xi, \qquad k^2 = \frac{z_1}{z_1 + |z_3|}, \qquad g = \frac{2}{\sqrt{z_1 + |z_3|}}$$

$$z_3 = 0, \quad 0 < z_2 < z_3$$

$$z_+ = z_1 - (z_1 - z_2)\mathsf{sn}^2 u \tag{24}$$

$$u = \frac{2\sqrt{\mu}}{g}\xi, \qquad k^2 = \frac{z_1 - z_2}{z_1}, \qquad g = \frac{2}{\sqrt{z_1}}$$

In the degenerate case, when $k \to 1$ ($\mathsf{cn}\ u \to \mathsf{sech}\ u, \quad \mathsf{sn}\ u \to \mathsf{tanh}\ u$), in both cases we get a bright soliton

$$z_+ \to z_1 \frac{1}{\mathsf{cosh}^2 u} \qquad u = \frac{2\sqrt{\mu}}{g}, \qquad g = \frac{2}{\sqrt{z_1}} \tag{25}$$

In conclusion, in this case of motion with constant velocities and equal constants $c_1 = c_2 = c_0$, the solutions are bright solitons. It is clear that in this case no energy transfer between the two components takes place.

As $v_i = \frac{d\theta_i}{dx} = v_0, \quad \theta_i(x,t) = v_0 x + \gamma_i(t)$ and using (9) the phase is easily calculated; one obtains

$$\theta_i = v_0 x - \left(\frac{1}{2}v_0^2 - \frac{E}{2}\right) t + \delta_i \tag{26}$$

Motion with stationary-profile current velocity

$$\rho_i(x,t) = \rho(\xi), \quad v_i(x,t) = v_i(\xi), \quad \xi = x - u_0 t \tag{27}$$

From the equations of continuity

$$v_i = u_0 + \frac{A_i}{\rho_i} \tag{28}$$

As in the previous section we consider $E_1 = E_2$. We get the same equations as in the previous section, but without any restriction on the solution. A larger class of periodic solutions can be considered. As an example we consider the case $0 < z_2 < z_1$ (no restriction on z_2 and z_3) when we get

$$z_+ = z_1 - (z_1 - z_2)\mathsf{sn}^2 u \tag{29}$$

$$u = \frac{2\sqrt{\mu}}{g}\xi, \qquad k^2 = \frac{z_1 - z_2}{z_1 - z_3}, \qquad g = \frac{2}{\sqrt{z_1 - z_2}}$$

The degenerate case $\lambda = 1$ is obtained for $z_2 = z_3 > 0$, and

$$z_+ = z_1 - (z_1 - z_2)\tanh^2 u \tag{30}$$

corresponding to a bright-type soliton with nonvanishing values at infinity (shifted-bright soliton).

The expression of the phase takes a more complicated form containing incomplete elliptic integral of third kind.

For $E_1 \neq E_2$ we shall use a direct method for solving the coupled system of equations ($i = 1, 2$)

$$\frac{1}{4}\frac{d^3\rho_i}{d\xi^3} - E_i\frac{d\rho_i}{d\xi} + 2\gamma(\rho_1 + \rho_2)\frac{d\rho_i}{d\xi} + \gamma\rho_i\frac{d}{d\xi}(\rho_1 + \rho_2) = 0.$$

We seek solutions of the form

$$\rho_i = A_i + B_i \mathsf{sn}^2 u, \qquad u = 2\lambda\xi.$$

Details are presented in [10]. In the degenerate case ($k^2 = 1$) we get

$$\rho_1 = \frac{4\lambda^2}{\gamma} b(\mu + 2\delta - \tanh^2 u)$$

$$\rho_2 = \frac{4\lambda^2}{\gamma}(1-b)(\mu - \tanh^2 u),$$

with $\mu > 1, \quad 0 < b < 1$, and $u = 2\lambda\xi$. They represent shifted-bright solitons.

Conclusions

A multiple scales analysis for three wave interaction in a multiple mode optical fiber is performed, assuming that one mode has normal dispersion and the other two anomalous dispersion. If a long wave-short wave interaction takes place a two component Zakharov-Yajima-Oikawa system results. This is analyzed using a Madelung fluid ansatz for the short waves. The resulting system is solved in two cases, namely 1) equal and constant velocities, and 2) motion with stationary profile current velocities. Expressions for the bright solitons in these cases are presented.

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