Two-dimensional lattice for four-dimensional supersymmetric Yang-Mills*

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Abstract

We discuss a lattice formulation for supersymmetric Yang-Mills (SYM) theories with extended supersymmetry, which preserves a part of supersymmetry on lattice.

For cases of two dimensions, we can see that lattice models in such a formulation lead to the target continuum theories with no fine-tuning. Namely, supersymmetries or some other symmetries not realized on the lattice are automatically restored in the continuum limit.

Next, we consider a mass deformation to $\mathcal{N} = (8, 8)$ SYM and present its lattice formulation with keeping two supersymmetries. It provides a nonperturbative framework to investigate IIA matrix string theory. Moreover, since it has fuzzy sphere solutions around which four-dimensional theory is deconstructed, it will serve a nonperturbative formulation of four-dimensional $\mathcal{N} = 4$ SYM which requires no fine-tuning. The rank of the gauge group is not restricted to large N. It opens a quite interesting possibility to test AdS/CFT correspondence in a stringy regime where string loop effects cannot be neglected.

Also, for two-dimensional $\mathcal{N} = (4, 4)$ SYM, a similar argument is possible to obtain four-dimensional $\mathcal{N} = 2$ SYM on noncommutative space.

1. Introduction

In this article, a lattice formulation of supersymmetric gauge theories is discussed. Lattice formulations mean to reformulate quantum field theories on discretized space-time, which makes possible computer simulations. They serve a conventional tool to investigate nonperturbative aspects of quantum field theories. It is a quite interesting issue to extend such formulations for supersymmetric gauge field theories, since they have been

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an intriguing candidate describing physics beyond the standard model and they have been found to have a close connection to string theory via matrix string conjectures [2] and AdS/CFT correspondence [3].

However, there is a notorious difficulty regarding a realization of supersymmetry on lattice. Typical supersymmetry algebra

$$\{Q_\alpha, Q_\beta\} \sim \gamma^\mu_{\alpha\beta} p_\mu \tag{1}$$

contains momenta p_μ which act as derivatives to continuous space-time. Naively, derivative operators are replaced with difference operators upon discretizing space-time to lattice. Note that the Leibniz rule, which holds for derivative operators, becomes broken for difference operators by the quantity $\mathcal{O}(a)$. Here, a is a lattice spacing. It generally leads to an explicit breaking of supersymmetry in a lattice action, which is naively constructed, by the amount $\mathcal{O}(a)$. So, near the continuum limit, i.e. for a being small, the lattice action S^{lat} can be written as the sum of the corresponding continuum action S^{cont} and the $\mathcal{O}(a)$ quantity $\tilde{S}$:

$$S^{\mathrm{lat}} = S^{\mathrm{cont}} + \tilde{S}. \tag{2}$$

As long as at the classical level, $\tilde{S}$ is irrelevant in the continuum limit ($a \to 0$). But, quantum mechanically, terms in $\tilde{S}$ may receive radiative corrections which become diverge as $a \to 0$. Note that a plays the role of a UV cutoff. Suppose a radiative correction from some term in $\tilde{S}$ behaves as $\mathcal{O}\left(\frac{1}{a^\sharp}\right)$. Then, the term becomes relevant when

$$\mathcal{O}(a) \times \mathcal{O}\left(\frac{1}{a^\sharp}\right) = \mathcal{O}\left(\frac{1}{a^{\sharp-1}}\right) \geq \mathcal{O}(a^0). \tag{3}$$

It means that supersymmetry breaking terms in $\tilde{S}$ can be relevant at the quantum level, and then we cannot reach the desired supersymmetric theory by sending $a \to 0$ naively in (2). In that case, in order to obtain the supersymmetric continuum theory, we should such supersymmetry breaking relevant operators in advance. This procedure is called as fine-tuning.

Since it is quite cumbersome to carry out fine-tuning in actual computer simulations, lattice models requiring no fine-tuning are practically best. So, let us focus on cases that lattice actions have "good symmetries" which protect from generating supersymmetry breaking radiative corrections. For some theories with extended supersymmetry, it has been found that not all supersymmetries, but some part of "nilpotent" supersymmetry not generating translations plays a role of the "good symmetries". Some of such examples are

- $\mathcal{N} = (2,2)$ supersymmetric Yang-Mills (SYM) theory in two dimensions [4, 5, 6]
- $\mathcal{N} = (2,2)$ supersymmetric QCD (SQCD) in two dimensions [7, 8, 9, 10, 11]

- $\mathcal{N} = (4,4)$ SYM in two dimensions [6, 12].

In what follows, we present a new lattice formulation for two-dimensional $\mathcal{N} = (8,8)$ SYM theory with $U(N)$ gauge group. By using a plane-wave like mass deformation, flat directions of scalar fields are lifted with preserving two "nilpotent" supercharges. Next, we discuss a scenario to obtain four-dimensional $\mathcal{N} = 4$ SYM from a fuzzy S^2 background of the two-dimensional theory. Note that the gauge group of the four-dimensional theory is $U(k)$ with k being arbitrary. As related work, [13] leads to three-dimensional $\mathcal{N} = 8$ SYM from a fuzzy S^2 background in the BMN matrix model which is a matrix quantum mechanics [14]. Also, [15] obtains four-dimensional $\mathcal{N} = 4$ SYM on $\mathbf{R} \times S^3$ with the gauge group $U(\infty)$, i.e. a planar gauge theory, from the BMN matrix model.

2. Continuum 2d $\mathcal{N} = (8,8)$ SYM

Let us start with the Euclidean action of $\mathcal{N} = (8,8)$ SYM in continuum two-dimensional space-time:

$$S_0 = \frac{2}{g_{2d}^2} \int d^2x \, \mathrm{tr} \left[\frac{1}{2} F_{12}^2 + \frac{1}{2}(D_\mu X^I)^2 - \frac{1}{4}[X^I, X^J]^2 \right. \\ \left. + \frac{1}{2}\Psi^T(D_1 + \gamma_2 D_2)\Psi + \frac{i}{2}\Psi^T \gamma_I [X^I, \Psi] \right], \qquad (4)$$

where $\mu = 1, 2$, and $I, J = 3, 4, \cdots, 10$. $F_{12} = \partial_1 A_2 - \partial_2 A_1 + i[A_1, A_2]$ is a gauge field strength, and $D_\mu = \partial_\mu + i[A_\mu, \cdot]$ gauge covariant derivatives. X^I $(I = 3, 4, \cdots, 10)$ represent 8 adjoint scalars, and Ψ 16 component adjoint fermions. $\gamma_{I'}$ $(I' = 2, 3, \cdots, 10)$ are imaginary symmetric 16×16 matrices satisfying $\{\gamma_{I'}, \gamma_{J'}\} = -2i\delta_{I'J'} \mathbb{1}_{16}$.

It is convenient to rewrite (4) in the language of topological twist, leading it to the form of balanced topological field theory [16]. The 8 scalars X^I and the 16 components of Ψ are denoted as

$$X^I \Longrightarrow \begin{cases} X_i & (i = 3, 4) \\ B_{\mathtt{A}} & (\mathtt{A} = 1, 2, 3) \\ C, \, \phi_+, \, \phi_-, & \end{cases} \qquad (5)$$

$$\Psi \Longrightarrow \begin{cases} \psi_{+\mu}, \, \rho_{+i}, \, \chi_{+\mathtt{A}}, \, \eta_+ \\ \psi_{-\mu}, \, \rho_{-i}, \, \chi_{-\mathtt{A}}, \, \eta_-. \end{cases} \qquad (6)$$

$\phi_\pm$ are complex matrices,which are hermitian conjugate with each other. There are appropriate two supercharges $Q_+^{(0)}, Q_-^{(0)}$ to recast the action to the $Q_+^{(0)}Q_-^{(0)}$ exact form [1]:

$$S_0 = Q_+^{(0)}Q_-^{(0)}\mathcal{F}_0, \tag{7}$$

$$\mathcal{F}_0 = \frac{1}{g_{2d}^2}\int d^2x\,\mathrm{tr}\left[-iB_{\mathtt{A}}\Phi_{\mathtt{A}} - \frac{1}{3}\epsilon_{\mathtt{ABC}}B_{\mathtt{A}}[B_{\mathtt{B}}, B_{\mathtt{C}}] \right.$$
$$\left. -\psi_{+\mu}\psi_{-\mu} - \rho_{+i}\rho_{-i} - \chi_{+\mathtt{A}}\chi_{-\mathtt{A}} - \frac{1}{4}\eta_+\eta_-\right] \tag{8}$$

with

$$\Phi_1 = 2(-D_1X_3 - D_2X_4), \qquad \Phi_2 = 2(-D_1X_4 + D_2X_3),$$
$$\Phi_3 = 2(-F_{12} + i[X_3, X_4]). \tag{9}$$

The supercharges $Q_\pm^{(0)}$ act as

$$Q_\pm^{(0)}A_\mu = \psi_{\pm\mu}, \quad Q_\pm^{(0)}\psi_{\pm\mu} = \pm iD_\mu\phi_\pm, \quad Q_\mp^{(0)}\psi_{\pm\mu} = \frac{i}{2}D_\mu C \mp \tilde{H}_\mu,$$
$$Q_\pm^{(0)}\tilde{H}_\mu = [\phi_\pm, \psi_{\mp\mu}] \mp \frac{1}{2}[C, \psi_{\pm\mu}] \mp \frac{i}{2}D_\mu\eta_\pm \tag{10}$$

to gauge multiplets $(A_\mu, \psi_{\pm\mu}, \tilde{H}_\mu)$ with $\tilde{H}_\mu$ auxiliary fields,

$$Q_\pm^{(0)}X_i = \rho_{\pm i}, \quad Q_\pm^{(0)}\rho_{\pm i} = \mp[X_i, \phi_\pm], \quad Q_\mp^{(0)}\rho_{\pm i} = -\frac{1}{2}[X_i, C] \mp \tilde{h}_i,$$
$$Q_\pm^{(0)}\tilde{h}_i = [\phi_\pm, \rho_{\pm i}] \mp \frac{1}{2}[C, \rho_{\pm i}] \pm \frac{1}{2}[X_i, \eta_\pm] \tag{11}$$

to X_i multiplets $(X_i, \rho_{\pm i}, \tilde{h}_i)$ with $\tilde{h}_i$ auxiliary fields,

$$Q_\pm^{(0)}B_{\mathtt{A}} = \chi_{\pm\mathtt{A}}, \quad Q_\pm^{(0)}\chi_{\pm\mathtt{A}} = \pm[\phi_\pm, B_{\mathtt{A}}], \quad Q_\mp^{(0)}\chi_{\pm\mathtt{A}} = \frac{1}{2}[C, B_{\mathtt{A}}] \mp H_{\mathtt{A}},$$
$$Q_\pm^{(0)}H_{\mathtt{A}} = [\phi_\pm, \chi_{\mp\mathtt{A}}] \pm \frac{1}{2}[B_{\mathtt{A}}, \eta_\pm] \mp \frac{1}{2}[C, \chi_{\pm\mathtt{A}}] \tag{12}$$

to $B_{\mathtt{A}}$ multiplets $(B_{\mathtt{A}}, \chi_{\pm\mathtt{A}}, H_{\mathtt{A}})$ with $H_{\mathtt{A}}$ auxiliary fields, and

$$Q_\pm^{(0)}C = \eta_\pm, \quad Q_\pm^{(0)}\eta_\pm = \pm[\phi_\pm, C], \quad Q_\mp^{(0)}\eta_\pm = \mp[\phi_+, \phi_-],$$
$$Q_\pm^{(0)}\phi_\pm = 0, \quad Q_\mp^{(0)}\phi_\pm = \mp\eta_\pm \tag{13}$$

to parameter multiplets $(C, \phi_\pm, \eta_\pm)$.

[1] This is essentially obtained by dimensional reduction from eq. (4.12) in [17].

From the above transformation property, we find that $Q_\pm^{(0)}$ are "nilpotent" in the sense that

$$
\begin{aligned}
(Q_+^{(0)})^2 &= (\text{infinitesimal gauge transformation by } \phi_+), \\
(Q_-^{(0)})^2 &= (\text{infinitesimal gauge transformation by } -\phi_-), \\
\{Q_+^{(0)}, Q_-^{(0)}\} &= (\text{infinitesimal gauge transformation by } C).
\end{aligned}
\tag{14}
$$

Thus, S_0 in (8) is manifestly invariant under $Q_\pm^{(0)}$ supersymmetries. Note that $\mathcal{F}_0$ is gauge invariant. There is an $SU(2)_R$ symmetry as another manifest symmetry of S_0, which is a subgroup of $SO(8)$ internal symmetry of the theory. Its generators J_{++}, J_{--}, J_0, expressed as

$$
\begin{aligned}
J_{\pm\pm} = \int d^2x \Bigg[& \psi^\alpha_{\pm\mu}(x)\frac{\delta}{\delta\psi^\alpha_{\mp\mu}(x)} + \chi^\alpha_{\pm\mathtt{A}}(x)\frac{\delta}{\delta\chi^\alpha_{\mp\mathtt{A}}(x)} - \eta^\alpha_\pm(x)\frac{\delta}{\delta\eta^\alpha_\mp(x)} \\
& \pm 2\phi^\alpha_\pm(x)\frac{\delta}{\delta C^\alpha(x)} \mp C^\alpha(x)\frac{\delta}{\delta\phi^\alpha_\mp(x)}\Bigg], \\
J_0 = \int d^2x \Bigg[& \psi^\alpha_{+\mu}(x)\frac{\delta}{\delta\psi^\alpha_{+\mu}(x)} - \psi^\alpha_{-\mu}(x)\frac{\delta}{\delta\psi^\alpha_{-\mu}(x)} + \chi^\alpha_{+\mathtt{A}}(x)\frac{\delta}{\delta\chi^\alpha_{+\mathtt{A}}(x)} \\
& - \chi^\alpha_{-\mathtt{A}}(x)\frac{\delta}{\delta\chi^\alpha_{-\mathtt{A}}(x)} + \eta^\alpha_+(x)\frac{\delta}{\delta\eta^\alpha_+(x)} - \eta^\alpha_-(x)\frac{\delta}{\delta\eta^\alpha_-(x)} \\
& + 2\phi^\alpha_+(x)\frac{\delta}{\delta\phi^\alpha_+(x)} - 2\phi^\alpha_-(x)\frac{\delta}{\delta\phi^\alpha_-(x)}\Bigg]
\end{aligned}
\tag{15}
$$

with $\alpha = 1, \cdots, N^2$ labeling a basis of $U(N)$ gauge group generators, satisfy

$$
[J_0, J_{\pm\pm}] = \pm 2J_{\pm\pm}, \qquad [J_{++}, J_{--}] = J_0. \tag{16}
$$

J_0 is a generator of $U(1)_R$ rotation, which is contained in $SU(2)_R$ as its Cartan subalgebra. Under the $SU(2)_R$ rotation, each of $(\psi_{+\mu}, \psi_{-\mu})$, $(\chi_{+\mathtt{A}}, \chi_{-\mathtt{A}})$, and $(\eta_+, -\eta_-)$ transforms as a doublet, and $(\psi_+, C, -\phi_-)$ as a triplet. Note that a pair of the supercharges $(Q_+^{(0)}, Q_-^{(0)})$ also forms a doublet. In particular,

$$
[J_{\pm\pm}, Q_\pm^{(0)}] = 0, \qquad [J_{\pm\pm}, Q_\mp^{(0)}] = Q_\pm^{(0)}. \tag{17}
$$

Fermions with the suffix $\pm$ carry the J_0-charge ± 1, and $\phi_\pm$ carry ± 2. Since $\mathcal{F}_0$ is invariant under the $SU(2)_R$, so is

$$
S_0 = Q_+^{(0)}Q_-^{(0)}\mathcal{F}_0 = \epsilon_{\gamma\delta}Q_\gamma^{(0)}Q_\delta^{(0)}\mathcal{F}_0 \qquad (\gamma, \delta = \pm). \tag{18}
$$

3. Mass deformation of 2d $\mathcal{N} = (8, 8)$ SYM

Next, we introduce mass terms to the theory presented in the previous section with preserving two supercharges. The $Q^{(0)}_\pm$-transformation (10)-(13) itself is deformed by a mass parameter M as

$$Q_\pm = Q^{(0)}_\pm + \Delta Q_\pm \tag{19}$$

with nontrivial transformation of $\Delta Q_\pm$ given by

$$\begin{aligned}
&\Delta Q_\pm \tilde{H}_\mu = \frac{M}{3}\psi_{\pm\mu}, \qquad \Delta Q_\pm \tilde{h}_i = \frac{M}{3}\rho_{\pm i}, \qquad \Delta Q_\pm H_{\mathtt{A}} = \frac{M}{3}\chi_{\pm\mathtt{A}}, \\
&\Delta Q_\pm \eta_\pm = \frac{2M}{3}\phi_\pm, \qquad \Delta Q_\mp \eta_\pm = \pm\frac{M}{3}C.
\end{aligned} \tag{20}$$

Then, the deformed supercharges $Q_\pm$ remain to be "nilpotent" up to gauge and $SU(2)_R$ transformations:

$$\begin{aligned}
Q_+^2 &= (\text{infinitesimal gauge transformation by } \phi_+) + \frac{M}{3}J_{++}, \\
Q_-^2 &= (\text{infinitesimal gauge transformation by } -\phi_-) - \frac{M}{3}J_{--}, \\
\{Q_+, Q_-\} &= (\text{infinitesimal gauge transformation by } C) - \frac{M}{3}J_0.
\end{aligned} \tag{21}$$

Note that (Q_+, Q_-) is still a doublet under the $SU(2)_R$.

The mass deformed action we consider is

$$S_M = \left(Q_+Q_- - \frac{M}{3}\right)\mathcal{F}_M, \qquad \mathcal{F}_M = \mathcal{F}_0 + \Delta F, \tag{22}$$

$$\Delta\mathcal{F} = \frac{1}{g_{2d}^2}\int d^2x\,\mathrm{tr}\left[\sum_{\mathtt{A}=1}^{3}\frac{a_{\mathtt{A}}}{2}B_{\mathtt{A}}^2 + \sum_{i=3}^{4}\frac{c_i}{2}X_i^2\right]. \tag{23}$$

When coefficients $a_{\mathtt{A}}, c_i$ all fall into the interval $(-\frac{2M}{3}, 0)$, scalars $X_i, B_{\mathtt{A}}$ have positive mass terms. For convenience, let us take

$$a_1 = a_2 = a_3 = -\frac{2M}{9}, \qquad c_3 = c_4 = -\frac{4M}{9} \tag{24}$$

in the followings. Because the similar relations to (17) hold for the deformed supercharges, S_M is manifestly invariant under the $Q_\pm$ transformations:

$$Q_\pm S_M = 0. \tag{25}$$

The explicit form of the action is expressed as

$$S_M = S_0 + \Delta S, \tag{26}$$

where

$$\begin{aligned}\Delta S = \frac{1}{g_{2d}^2}\int d^2x \,\mathrm{tr}\Bigg[&\frac{2M^2}{81}\left(B_A^2 + X_i^2\right) + \frac{M^2}{9}\left(\frac{C^2}{4} + \phi_+\phi_-\right)\\ &- \frac{M}{2}C[\phi_+,\phi_-] + \frac{2M}{3}\,\psi_{+\mu}\psi_{-\mu} + \frac{2M}{9}\rho_{+i}\rho_{-i} + \frac{4M}{9}\,\chi_{+A}\chi_{-A}\\ &- \frac{M}{6}\,\eta_+\eta_- - \frac{4iM}{9}B_3\left(F_{12} + i[X_3,X_4]\right)\Bigg].\end{aligned} \tag{27}$$

The first line and the third line in (27) give mass terms to scalars and to fermions, respectively. The second line represents the so called Myers term [18]. Thanks to these terms, a fuzzy S^2 configuration satisfying

$$\begin{aligned}&[\phi_+,\phi_-] = \frac{M}{3}C, \qquad [C,\phi_\pm] = \pm\frac{2M}{3}\phi_\pm,\\ &B_{\mathtt{A}} = X_i = 0\end{aligned} \tag{28}$$

gives the minimum of the action ($S_M = 0$) preserving the $Q_\pm$ supersymmetries. Note that the contribution of the last line in (27) is not real, but pure imaginary. Also, we should recognize that the mass-deformed action preserves only two supersymmetries ($Q_\pm$) but the other 14 supersymmetries are broken by the deformation. So, the action may look somewhat similar to the action of PP wave matrix strings preserving 8 supersymmetries [19], but it is different.

4. Lattice formulation of mass deformed theory

Now we present a lattice formulation for the mass-deformed theory in the previous section. On the two-dimensional square lattice $\mathbf{Z}^2$, gauge fields are promoted to $U(N)$ group variables:

$$A_\mu(x) \quad \Longrightarrow \quad U_\mu(x) = e^{iaA_\mu(x)}, \tag{29}$$

where $U_\mu(x)$ is defined on the link $(x, x+\hat{\mu})$ which connects the lattice sites x and $x+\hat{\mu}$. $\hat{\mu}$ is a unit vector along the μ-th direction on the lattice. All the other fields are put on sites. Lattice fields are dimensionless and they are related to the continuum counterparts by

$$\begin{aligned}&(\text{scalars})^{\text{lat}} = a(\text{scalars})^{\text{cont}}, \qquad (\text{fermions})^{\text{lat}} = a^{3/2}(\text{fermions})^{\text{cont}},\\ &Q_\pm^{\text{lat}} = a^{1/2}Q_\pm^{\text{cont}}.\end{aligned} \tag{30}$$

Also, dimensionless coupling constants on the lattice are

$$g_0 = ag_{2d}, \qquad M_0 = aM. \tag{31}$$

$Q_\pm$ supersymmetries can be realized on the lattice as

$$\begin{aligned}
Q_\pm U_\mu(x) &= i\psi_{\pm\mu}(x)U_\mu(x),\\
Q_\pm \psi_{\pm\mu}(x) &= i\psi_{\pm\mu}(x)\psi_{\pm\mu}(x) \pm iD_\mu\phi_\pm(x),\\
Q_\mp \psi_{\pm\mu}(x) &= \frac{i}{2}\{\psi_{+\mu}(x), \psi_{-\mu}(x)\} + \frac{i}{2}D_\mu C(x) \mp \tilde{H}_\mu(x),\\
Q_\pm \tilde{H}_\mu(x) &= -\frac{1}{2}\left[\psi_{\mp\mu}(x), \phi_\pm(x) + U_\mu(x)\phi_\pm(x+\hat{\mu})U_\mu(x)^\dagger\right]\\
&\quad \pm\frac{1}{4}\left[\psi_{\pm\mu}(x), C(x) + U_\mu(x)C(x+\hat{\mu})U_\mu(x)^\dagger\right]\\
&\quad \mp\frac{i}{2}D_\mu\eta_\pm(x) \pm \frac{1}{4}\left[\psi_{\pm\mu}(x)\psi_{\pm\mu}(x), \psi_{\mp\mu}(x)\right]\\
&\quad +\frac{i}{2}\left[\psi_{\pm\mu}(x), \tilde{H}_\mu(x)\right] + \frac{M_0}{3}\psi_{\pm\mu},
\end{aligned} \tag{32}$$

where $D_\mu A(x) \equiv U_\mu(x)A(x+\hat{\mu})U_\mu(x)^{-1} - A(x)$ are forward covariant differences for adjoint fields $A(x)$. The transformation rules for the other fields are of the same form as in the continuum with the trivial replacement $M \to M_0$. Then, the "nilpotency" (21) is realized on the lattice.

In order to construct the corresponding lattice action, let us define a lattice counterpart of $\Phi_{\mathtt{A}}$ in (9) as

$$\begin{aligned}
\Phi_1(x) &= 2\left(-D_1X_3(x) - D_2X_4(x)\right),\\
\Phi_2(x) &= 2\left(-D_1^*X_4(x) + D_2^*X_3(x)\right),\\
\Phi_3(x) &= \frac{i(U_{12}(x) - U_{21}(x))}{1 - \epsilon^{-2}||1 - U_{12}(x)||^2} + 2i[X_3(x), X_4(x)].
\end{aligned} \tag{33}$$

Here, $D_\mu^* A(x) \equiv A(x) - U_\mu(x-\hat{\mu})^{-1}A(x-\hat{\mu})U_\mu(x-\hat{\mu})$ are backward covariant differences,

$$U_{\mu\nu}(x) = U_\mu(x)U_\nu(x+\hat{\mu})U_\mu(x+\hat{\nu})^{-1}U_\nu(x)^{-1} \tag{34}$$

are plaquette variables, and we take $||A|| = \sqrt{\mathrm{tr}(AA^\dagger)}$ as the norm of an arbitrary matrix A. ϵ is a positive number chosen as $0 < \epsilon < 2$ for the gauge group $U(N)$. The first term of the RHS of $\Phi_3(x)$ is a lattice counterpart of the field strength F_{12}. It is the same as the situation in the lattice formulation for two-dimensional $\mathcal{N} = (2,2)$ $U(N)$ SYM [6]. Then, $Q_\pm$-invariant lattice action is given as

$$S_{\mathrm{lat}} = \left(Q_+Q_- - \frac{M_0}{3}\right)\mathcal{F}_{\mathrm{lat}} \tag{35}$$

with $\mathcal{F}_{\rm lat}$ being the same form as $\mathcal{F}_M$ in (22) under the trivial replacement

$$\frac{1}{g_{2d}^2}\int d^2x \to \frac{1}{g_0^2}\sum_x, \qquad M \to M_0, \tag{36}$$

when the admissibility condition $||1 - U_{12}(x)|| < \epsilon$ is satisfied for $\forall x$. Otherwise,

$$S_{\rm lat} = +\infty, \tag{37}$$

i.e. the Boltzmann weight $e^{-S_{\rm lat}}$ vanishes.

4.1. Minimum of the lattice action

Here, we will check that the lattice action has the minimum only at the pure gauge configuration $U_{12}(x) = \mathbb{1}_N$, which guarantees that the weak field expansion

$$U_\mu(x) = 1 + iaA_\mu(x) + \frac{(ia)^2}{2!}A_\mu(x)^2 + \cdots \tag{38}$$

is allowed in the continuum limit and that the lattice theory converges to the desired continuum theory at the classical level.

After integrating out the auxiliary fields, bosonic part of the action $S_{\rm lat}$ takes the form

$$\begin{aligned} S_{\rm lat}^{(B)} = \frac{1}{g_0^2}\sum_x {\rm tr}\Bigg[&\frac{2M_0^2}{81}\left(X_i(x)^2 + B_{\tt A}(x)^2\right) \\ &- i\frac{4M_0}{9}B_3(x)\left\{-\frac{1}{2}\frac{i(U_{12}(x) - U_{21}(x))}{1 - \frac{1}{\epsilon^2}||1 - U_{12}(x)||^2} + i[X_3(x), X_4(x)]\right\}\Bigg] \\ &+ S_{\rm PDT} \end{aligned} \tag{39}$$

with $S_{\rm PDT}$ representing positive semi-definite terms. We will treat the terms in the second line in (39), which is pure imaginary, as operators by employing the reweighting method. Then, for the remaining part of $S_{\rm lat}^{(B)}$ we find that the mass terms in the first line fix the minimum at

$$X_i(x) = B_{\tt A}(x) = 0, \tag{40}$$

which is independent of $S_{\rm PDT}$. At (40), $S_{\rm PDT}$ becomes

$$\begin{aligned} S_{\rm PDT} = \frac{1}{g_0^2}\sum_x {\rm tr}\left[\sum_\mu (D_\mu X_p(x))^2 + \left(i[X_p(x), X_q(x)] + \frac{M_0}{3}\epsilon_{pqr}X_r(x)\right)^2\right] \\ + \frac{1}{4g_0^2}\sum_x \frac{{\rm tr}\left[-(U_{12}(x) - U_{21}(x))^2\right]}{\left(1 - \frac{1}{\epsilon^2}||1 - U_{12}(x)||^2\right)^2} \end{aligned} \tag{41}$$

with $C = 2X_8$, $\phi_{\pm} = X_9 \pm iX_{10}$ and $p, q, r = 8, 9, 10$. Notice that the last term representing the gauge kinetic term has the same form as in the case of two-dimensional $\mathcal{N} = (2,2)$ SYM [6], where the admissibility condition with $0 < \epsilon < 2$ for the gauge group $U(N)$ singles out the minimum $U_{12}(x) = 1\!\!1_N$. In order to illustrate it, suppose we forget the admissibility condition $||1 - U_{12}(x)|| < \epsilon$ for a moment. The action has the minimum at configurations satisfying

$$U_{12}(x)^2 + U_{21}(x)^2 = 2, \tag{42}$$

namely

$$U_{12}(x) = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} \tag{43}$$

up to gauge transformations with all the combinations of ± 1 in the diagonal entries allowed. Then, we should take into account fluctuations around each configuration of (43) although only the single configuration $U_{12}(x) = 1\!\!1_N$ allows the weak field expansion (38) leading to the target continuum theory. However, if the admissibility condition $||1 - U_{12}(x)|| < \epsilon$ is imposed, all the configurations of (43) are excluded except $U_{12}(x) = 1\!\!1_N$. In this way, we can successfully single out the configuration $U_{12}(x) = 1\!\!1_N$. Notice that the standard Wilson's lattice gauge action has the minimum at

$$U_{12}(x) + U_{21}(x) = 2 \tag{44}$$

leading to $U_{12}(x) = 1\!\!1_N$. "The square" of $U_{12}(x)$ or $U_{21}(x)$ in (42) is a crucial difference from the Wilson case (44).

Hence, we obtain the single minimum $U_{12}(x) = 1\!\!1_N$ for the action S_{lat}. The mass deformation preserving $Q_{\pm}$ is crucial to stabilize flat directions of scalars as well as to remove degeneracy of gauge fields.

4.2. Absence of doubler modes

In order to check that no doubler appears in the lattice action S_{lat}, let us set $U_{\mu}(x) = 1\!\!1_N$ and pick up quadratic kinetic terms.

Then, we have the kinetic terms for scalars

$$\begin{aligned} S_{\text{lat}}^{(2,\,B)} = \frac{1}{g_0^2} \sum_x \text{tr} \Bigg[& \frac{2M_0^2}{81} \left(\sum_i X_i(x)^2 + \sum_{\mathtt{A}} B_{\mathtt{A}}(x)^2 \right) \\ & + \sum_{\mu} \left\{ \Delta_{\mu}\phi(x) \Delta_{\mu}\bar{\phi}(x) + \frac{1}{4} \left(\Delta_{\mu} C(x) \right)^2 \right\} \\ & + \frac{M_0^2}{9} \left(\frac{1}{4} C(x)^2 + \phi(x)\bar{\phi}(x) \right) \end{aligned}$$

$$
\begin{aligned}
&+ (\Delta_1^* B_1(x) - \Delta_2 B_2(x))^2 + (\Delta_2^* B_1(x) + \Delta_1 B_2(x))^2 \\
&+ (\Delta_1 X_3(x) + \Delta_2 X_4(x))^2 + (-\Delta_1^* X_4(x) + \Delta_2^* X_3(x))^2 \\
&\left. + \sum_\mu \left(\Delta_\mu^* B_3(x)\right)^2 \right], \qquad (45)
\end{aligned}
$$

where Δ_μ (Δ_μ^*) are forward (backward) difference operators:

$$
\Delta_\mu f(x) = f(x+\hat{\mu}) - f(x), \qquad \Delta_\mu^* f(x) = f(x) - f(x-\hat{\mu}). \qquad (46)
$$

Using the relation

$$
\sum_x f(x) \Delta_\mu g(x) = -\sum_x \left(\Delta_\mu^* f(x)\right) g(x), \qquad (47)
$$

we find that (45) takes the form of the standard kinetic terms for scalars:

$$
\begin{aligned}
S_{\rm lat}^{(2,\,B)} = \frac{1}{g_0^2} \sum_x \mathrm{tr} \Bigg[& \sum_\mu \Delta_\mu \phi(x) \Delta_\mu \bar{\phi}(x) + \frac{M_0^2}{9} \phi(x) \bar{\phi}(x) \\
& + \frac{1}{4} \left\{ \sum_\mu (\Delta_\mu C(x))^2 + \frac{M_0^2}{9} C(x)^2 \right\} \\
& + \sum_{\mathtt{A}} \left\{ \sum_\mu (\Delta_\mu B_{\mathtt{A}}(x))^2 + \frac{2M_0^2}{81} B_{\mathtt{A}}(x)^2 \right\} \\
& + \sum_i \left\{ \sum_\mu (\Delta_\mu X_i(x))^2 + \frac{2M_0^2}{81} X_i(x)^2 \right\} \Bigg], \qquad (48)
\end{aligned}
$$

indicating that no doubler appears in the bosonic sector.
For fermions, the kinetic terms can be expressed in the form:

$$
\begin{aligned}
S_{\rm lat}^{(2,\,F)} = \frac{1}{g_0^2} \sum_x \mathrm{tr} \Bigg[& \sum_\mu \Psi^{(0)}(x)^T G_\mu \frac{1}{2} (\Delta_\mu + \Delta_\mu^*) \Psi^{(0)}(x) \\
& + \sum_\mu \Psi^{(0)}(x)^T P_\mu \frac{1}{2} (\Delta_\mu - \Delta_\mu^*) \Psi^{(0)}(x) \\
& + \Psi^{(0)}(x)^T \mathcal{M} \Psi^{(0)}(x) \Bigg]. \qquad (49)
\end{aligned}
$$

Here,

$$
\begin{aligned}
(\Psi^{(0)})^T = \Big(& \rho_{+3}, \rho_{+4}, \psi_{+2}, \psi_{+1}, -\chi_{+1}, \chi_{+2}, \chi_{+3}, \frac{1}{2}\eta_+, \\
& \rho_{-3}, \rho_{-4}, \psi_{-2}, \psi_{-1}, -\chi_{-1}, \chi_{-2}, \chi_{-3}, \frac{1}{2}\eta_- \Big). \qquad (50)
\end{aligned}
$$

G_μ are 16×16 imaginary symmetric (anti-hermitian) matrices and P_μ are 16×16 imaginary anti-symmetric (hermitian) matrices, satisfying

$$\{G_\mu, G_\nu\} = -2\delta_{\mu\nu}\mathbb{1}_{16}, \quad \{P_\mu, P_\nu\} = 2\delta_{\mu\nu}\mathbb{1}_{16}, \quad \{G_\mu, P_\nu\} = 0. \tag{51}$$

The mass matrix is

$$\mathcal{M} = \frac{M_0}{9}\begin{pmatrix} & m_d \\ -m_d & \end{pmatrix} \tag{52}$$

with

$$m_d = \mathrm{diag}\,(1,1,3,3,2,2,2,-3)\,. \tag{53}$$

When we move to the momentum space via

$$\Psi^{(0)}(x) = \int_{-\pi/a}^{\pi/a} \frac{d^2p}{(2\pi)^2}\,\tilde{\Psi}^{(0)}(p)e^{iap\cdot x}, \tag{54}$$

the kernel

$$\mathcal{D}_F = \sum_\mu \left[G_\mu \frac{1}{2}(\Delta_\mu + \Delta_\mu^*) + P_\mu \frac{1}{2}(\Delta_\mu - \Delta_\mu^*)\right] \tag{55}$$

is expressed as

$$\tilde{\mathcal{D}}_F(p) = \sum_{\mu=1}^{2}\left[iG_\mu \sin(ap_\mu) - 2P_\mu \sin^2\left(\frac{ap_\mu}{2}\right)\right]. \tag{56}$$

Using (51), we can easily see

$$\tilde{\mathcal{D}}_F(p)^2 = \sum_{\mu=1}^{2} 4\sin^2\left(\frac{ap_\mu}{2}\right). \tag{57}$$

Since $\tilde{\mathcal{D}}_F(p)$ is hermitian, (57) shows that only the origin $(p_1, p_2) = (0,0)$ gives the zero of $\tilde{\mathcal{D}}_F(p)$. It proves that there appears no doubler in the fermionic sector.

4.3. No need of fine-tuning

We give a perturbative argument to show that the lattice action converges to the desired continuum theory in the quantum mechanical sense without any fine-tuning.

After integrating out the auxiliary fields in the theory near the continuum limit, let us consider local operators of the type:

$$\mathcal{O}_p(x) = M^m \varphi(x)^\alpha \partial^\beta \psi(x)^{2\gamma}, \tag{58}$$

where $m, \alpha, \beta, \gamma = 0, 1, 2, \cdots$, and ∂ means derivative operators. $\varphi(x)$ and $\psi(x)$ represent a general bosonic field and a general fermionic field, respectively. The mass dimension of $\mathcal{O}_p(x)$ is $p \equiv m + \alpha + \beta + 3\gamma$. Dimensional analysis tells that radiative corrections to $\mathcal{O}_p(x)$ has the form

$$\left(\frac{1}{g_{2d}^2} c_0 a^{p-4} + c_1 a^{p-2} + g_{2d}^2 c_2 a^p + \cdots\right) \int d^2x \, \mathcal{O}_p(x) \tag{59}$$

up to possible powers of $\ln(aM)$. c_1, c_2, c_3 are dimensionless numerical constants. The first, second and third terms in the parenthesis are contributions from tree, 1-loop and 2-loop effects, respectively. The "$\cdots$" is a contribution from higher loops, which is irrelevant for the analysis. Since relevant or marginal operators generated by loop effects possibly appear from nonpositive powers of a in the second and third terms, we should see operators with $p = 0, 1, 2$. They are φ, $M\varphi$ and φ^2. (Note that $\mathbb{1}$, M, M^2 and $\partial\varphi$ are not dynamical.) Candidates for φ are linear combinations of $\mathrm{tr}X_i$ and $\mathrm{tr}B_{\mathtt{A}}$ from gauge and $SU(2)_R$ symmetries. But, all of them are not invariant under $Q_\pm$ supersymmetries, and thus are forbidden to appear. Similarly we can show that $M\varphi$ and φ^2 are not allowed to be generated. Hence, we can conclude that any relevant or marginal operators except nondynamical constant operators are not radiatively generated. It is shown that no fine-tuning is required in taking the continuum limit.

5. Matrix String theory

The mass-deformed $\mathcal{N} = (8, 8)$ SYM in two dimensions can be obtained from the lattice theory around the trivial minimum $C = \phi_\pm = 0$ as seen in the previous section. Since M is a soft mass breaking 16 supersymmetries to $Q_\pm$, the undeformed theory, which is nothing but IIA matrix string theory [2], can be defined by turning off M after the continuum limit.

6. 4d $\mathcal{N} = 4$ SYM

In this section, we discuss a scenario to obtain four-dimensional $\mathcal{N} = 4$ SYM from the lattice formulation given in section 4..

Let us consider the lattice theory around the minimum of k-coincident fuzzy S^2:

$$C = \frac{2M_0}{3} L_3, \qquad \phi_\pm = \frac{M_0}{3}(L_1 \pm iL_2) \tag{60}$$

with $L_a = L_a^{(n)} \otimes \mathbb{1}_k$ ($a = 1, 2, 3$) and $N = nk$. $L_a^{(n)}$ are $SU(2)$-generators of n-dimensional irreducible representation.

First, we take the continuum limit of the two-dimensional lattice theory. Then, we obtain four-dimensional $\mathcal{N} = 4$ $U(k)$ SYM on $\mathbf{R}^2 \times$ (Fuzzy S^2) with 16 supersymmetries softly broken to $Q_\pm$. Noncommutativity of the fuzzy S^2 is characterized by the parameter

$$\theta \simeq \frac{1}{M^2 n}, \tag{61}$$

and in fuzzy S^2 directions UV cutoff is set at $\Lambda \simeq Mn$ and IR cutoff is given by M.

Next, we consider the following two steps:

1. Take large n limit with θ and k fixed. Namely, $M \propto n^{-1/2} \to 0$ and $\lambda \propto n^{1/2} \to \infty$.
2. Send θ to zero.

6.1. Step 1

At the step 1, we should investigate radiative corrections in four-dimensional SYM on $\mathbf{R}^2 \times$(Fuzzy S^2). However, we give an argument below that there is no radiative correction which prevents from full 16 supersymmetries being restored after the step 1.

For a general quantum field theory defined on noncommutative (NC) space-time, we should consider two kinds of Feynman diagrams separately. One is planar diagrams which have no NC phase factors causing UV/IR mixing. Much is the same as in the theories on ordinary space-time, and M appearing in the planar diagrams is soft for UV singularities. The other is nonplanar diagrams. There appear NC phase factors which improve the UV behavior of the diagrams. But, IR singularities come from vanishing NC phase which arise even in massive theories (UV/IR mixing). Then, we can say that M is insensitive ("soft") for singularities from UV/IR mixing.

The superficial degrees of UV divergence of Feynman diagrams is given by

$$D = 4 - E_B - \frac{3}{2}E_F, \tag{62}$$

where E_B and E_F are the number of the external lines of bosons and that of fermions, respectively. Note that the divergence with $D = 3$ by $E_B = 1$ is absent since the operator φ is forbidden by $Q_\pm$ supersymmetries. Thus, possible most severe divergence $D = 2$ comes from UV divergences in planar diagrams and from UV/IR divergences in nonplanar diagrams. The leading Λ^2 terms are expected to cancel each other by 16 supersymmetries, because M is a soft mass and the full 16 supersymmetries are effective to the leading contribution in UV region. Possible divergences originate from the mass deformation, whose behavior is expected as

$$M^p \left(\ln \frac{\Lambda}{M}\right)^q \simeq M^p (\ln n)^q \qquad (p = 1, 2,\ q = 1, 2, \cdots) \tag{63}$$

which vanishes in the limit taken at the step 1.

If this argument works, no radiative correction prevents restoration of the full 16 supersymmetries after the step 1. Therefore, four-dimensional $\mathcal{N} = 4$ $U(k)$ SYM on $\mathbf{R}^2 \times$ NC $\mathbf{R}^2$ with 16 supersymmetries is obtained.

6.2. Step 2

In four-dimensional $\mathcal{N} = 4$ SYM on NC space, $\theta \to 0$ limit is believed to be smooth [20]. (See also [21] for discussions in the context of AdS/CFT correspondence.) If this is true, desired $U(k)$ SYM on ordinary $\mathbf{R}^4$ can be obtained with no fine-tuning after the step 2.

7. Summary and discussion

In this article, firstly we presented a lattice formulation for mass-deformed $\mathcal{N} = (8, 8)$ $U(N)$ SYM in two dimensions preserving two supercharges $Q_\pm$. Owing to the mass-deformation, flat directions of scalars are stabilized. Also, since the mass is soft, ordinary two-dimensional $\mathcal{N} = (8, 8)$ SYM is obtained by turning off the mass after the continuum limit. It is applicable to nonperturbative analysis of IIA matrix string theory [2].

Secondly, we discussed a scenario to obtain four-dimensional $\mathcal{N} = 4$ $U(k)$ SYM from the two-dimensional theory around a fuzzy S^2 background. To establish the scenario, explicit calculation of diagrams of the theory on $\mathbf{R}^2 \times (\text{Fuzzy } S^2)$ is desirable. If it is correct, four-dimensional $\mathcal{N} = 4$ SYM with gauge group $U(k)$ of finite rank can be defined nonperturbatively. It will give an intriguing tool to check AdS/CFT correspondence [3] in a regime where string loop effects cannot be neglected.

Thirdly, it is possible to construct a similar mass-deformed lattice model for two-dimensional $\mathcal{N} = (4, 4)$ $U(N)$ SYM. Then, flat directions of scalars are lifted preserving $Q_\pm$ supersymmetries. In particular, the mass-deformed theory preserves the full 8 supersymmetries. Four-dimensional theory constructed around a fuzzy S^2 background will become $\mathcal{N} = 2$ $U(k)$ SYM on $\mathbf{R}^2 \times$ NC $\mathbf{R}^2$. This time, the limit turning off θ will not be smooth.

Finally, it is also interesting to construct a similar lattice formulation for SQCD coupled with matter fields and for $\mathcal{N} = 1^*, 2^*$ theories.

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