Generalized Gell-Mann formula for $sl(n,\mathbb{R})$ and application examples*

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Abstract

The so called Gell-Mann (decontraction) formula, that provides an expression of the Lie algebra elements in terms of the elements of the Inönü-Wigner contracted algebra, is of limited validity. Generalization of this formula for $sl(n,\mathbb{R})$ algebras, $n \geq 2$, contracted w.r.t. the corresponding $so(n)$ subalgebras was recently achieved. The generalized formula holds for all representations: tensorial and spinorial, unitary and non unitary, finite and infinite, with or without $so(n)$ multiplicity, thus covering any possible physics application. Basic results concerning the generalized Gell-Mann formula and its interpretation, as well as a few of its physically interesting applications in the Affine gravity context, are presented.

1. Introduction

The Gell-Mann, or decontraction, formula is a prescription aimed to serve as an inverse to the procedure of Inönü-Wigner contraction. In some recent papers on Gell-Mann decontraction formula in the $sl(n,\mathbb{R})$ case, we established its domain of validity [1] and constructed a generalization of this formula that is valid for all representations [2, 3]. The generalized formula allows us to find matrix representations of $sl(n,\mathbb{R})$ operators in all representations, thus covering all possible physical applications. They are given in the basis od compact subgroup – which is suitable for physical interpretations. Important feature of the generalized Gell-Mann formula is

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that it directly provides generator representations also in the case of spinorial $\overline{SL}(n,\mathbb{R})$ representations (i.e. representations that contain spinorial $Spin(n)$ subrepresentations), as well as in the case of representations with nontrivial $Spin(n)$ multiplicity. These properties are essential for many physical applications that the original formula failed to cover.

In this paper we endeavor to illustrate, through a few simple examples, how the (generalized) Gell-Mann formula can be utilized in the realm of theory of gravity. In particular, we shall concentrate on Affine models of gravity, in which the space-time symmetry of the theory (prior to any symmetry breaking) is given by the General Affine Group $GA(n,\mathbb{R}) = T^n \wedge GL(n,\mathbb{R})$ (or, sometimes, by the Special Affine Group $SA(n,\mathbb{R}) = T^n \wedge SL(n,\mathbb{R})$). In the quantum case, the General Affine Group is replaced by its double cover counterpart $\overline{GA}(n,\mathbb{R}) = T^n \wedge \overline{GL}(n,\mathbb{R})$, which contains double cover of $\overline{GL}(n,\mathbb{R})$ as a subgroup. This subgroup here plays the role that Lorentz group has in the Poincaré symmetry case. Thus it is clear that knowledge of $\overline{GL}(n,\mathbb{R})$ representations is a must-know for any serious analysis of Affine Gravity models. On the other hand, the essential part of the $\overline{GL}(n,\mathbb{R}) = R_+ \otimes \overline{SL}(n,\mathbb{R})$ group is its $\overline{SL}(n,\mathbb{R})$ subgroup, and that is where $\overline{SL}(n,\mathbb{R})$ generator matrix elements, obtained by using the generalized Gell-Mann formula, come into play (R_+ is subgroup of dilatations). We will apply expression for these matrix elements in order to obtain coefficients for some of the gauge field–matter interaction vertices.

2. Gauge Affine action

A standard way to introduce interactions into Affine gravity models is by localization of the global Affine symmetry $\overline{GA}(n,\mathbb{R}) = T^n \wedge \overline{GL}(n,\mathbb{R})$. Thus, quite generally, Affine Lagrangian consists of a gravitational part (i.e. kinetic terms for gauge potentials) and Lagrangian of the matter fields: $L = L_g + L_m$. Gravitational part L_g is a function of gravitational gauge potentials and their derivatives, and also of the dilaton field φ (that ensures action invariance under local dilatations). In the case of the standard Metric Affine gravity [4, 5], gravitational potentials are tetrads $e^a{}_\mu$, metrics g_{ab} and Affine connection $\Gamma^a{}_{b\mu}$, so that we can write: $L_g = L_g(e, \partial e, g, \partial g, \Gamma, \partial\Gamma, \varphi)$. More precisely, due to action invariance under local Affine transformations, gravitational part of Lagrangian must be a function of the form $L_g = L_g(e, g, T, R, N, \varphi)$, where $T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu + \Gamma^a{}_{b\mu} e^b{}_\nu - (\mu \leftrightarrow \nu)$, $R^a{}_{b\mu\nu} = \partial_\mu \Gamma^a{}_{b\nu} + \Gamma^c{}_{b\mu}\Gamma^a{}_{c\nu} - (\mu \leftrightarrow \nu)$, $N_{\mu ab} = D_\mu g_{ab}$ are, respectively, torsion, curvature and nonmetricity. Assuming, as usual, that equations of motion are linear in second derivatives of gauge fields, we are confined to no higher than quadratic powers of the torsion, curvature and nonmetricity. Covariant derivative is of the form $D_\mu = \partial_\mu - i\Gamma_a{}^b{}_\mu Q_b{}^a$, where $Q_b{}^a$ denote generators of $\overline{GL}(n,\mathbb{R})$ group. The matter Lagrangian (assuming minimal coupling for all fields except the dilaton one) is a function of some number of Affine fields ϕ^I and their covariant derivatives, together with metrics and tetrads (Affine connection enters only through covariant derivative): $L_m = L_m(\phi^I, D\phi^I, e, g)$.

With all these general remarks, we will consider a class of Affine Lagrangians, in arbitrary number of dimensions n, of the form:

$$L(e_\mu{}^a, \partial_\nu e_\mu{}^a, \Gamma_{b\mu}{}^a, \partial_\nu \Gamma_{b\mu}{}^a, g_{ab}, \Psi_A, \partial_\nu \Psi_A, \Phi_A, \partial_\nu \Phi_A, \varphi, \partial_\nu \varphi)$$
$$= e\Big[\varphi^2 R - \varphi^2 T^2 - \varphi^2 N^2 + \bar{\Psi} i g^{ab} \gamma_a e_b{}^\mu D_\mu \Psi + \tfrac{1}{2}\, g^{ab} e_a{}^\mu e_b{}^\nu (D_\mu \Phi)^+ (D_\nu \Phi)$$
$$+ \tfrac{1}{2}\, g^{ab} e_a{}^\mu e_b{}^\nu D_\mu \varphi D_\nu \varphi - L_g(n) + L_m(n)\Big]. \tag{1}$$

The terms in the first row represent general gravitational part of the Lagrangian, that is invariant w.r.t. Affine transformations (dilatational invariance is obtained with the aid of field φ, of mass dimension $n/2-1$). Here T^2 and N^2 stand for linear combination of terms quadratic in torsion and nonmetricity, respectively, formed by irreducible components of these fields (a discussion of available possibilities can be found in Appendix B of [5]). For the scope of this paper, we need not fix these terms any further. This is a general form of gravitational kinetic terms, invariant for an arbitrary space-time dimension $n \geq 3$.

The Lagrangian matter terms, invariant w.r.t. the local $\overline{GA}(n,\mathbb{R})$, $n \geq 3$, transformations, are written in the second row. The field Ψ denotes a spinorial $\overline{GL}(n,\mathbb{R})$ field – components of that field transform under some appropriate spinorial $\overline{GL}(n,\mathbb{R})$ irreducible representations. All spinorial $\overline{GL}(n,\mathbb{R})$ representations are necessarily infinite dimensional [6], and thus the field Ψ will have infinite number of components. The concrete spinorial irreducible representation of field Ψ is given by a set of $n-1$ $\overline{SL}(n,\mathbb{R})$ labels $\{\sigma_c^\Psi\}$ together with the dilatation charge d_Ψ. The field Φ is a representative of a tensorial $\overline{GL}(n,\mathbb{R})$ field, transforming under a tensorial $\overline{GL}(n,\mathbb{R})$ representation (i.e. one transforming w.r.t. single-valued representation of the $SO(n)$ subgroup) labelled by parameters $\{\sigma_c^\Phi\}$ and d_Φ. Since, as it is argued in the following section, the noncompact $\overline{SL}(n-1,\mathbb{R})$ Affine subgroup is to be represented unitarily, the tensorial field Φ is also to transform under an infinite-dimensional representation and to have an infinite number of components. The remaining dilaton field φ is scalar with respect to $\overline{SL}(n,\mathbb{R})$ subgroup, and thus has only one component.

Finally, the third row contains possible additional gravitational and matter terms, denoted respectively by $L_g(n)$ and $L_m(n)$, that, due to restrictions imposed by the dilatational invariance requirement, can appear only for some concrete values of n. (E.g., in [8] dealing with the four dimensional case, authors take $L_g(4) = \alpha_1 R_{[abcd]} R^{[abcd]} + \alpha_2 R_{[ab[c]d]} R^{[ab[c]d]} + \alpha_3 R_{[a(b][c)d]} R^{[a(b][c)d]} + \alpha_4 R_{(a[b)cd]} R^{(a[b)cd]} + \alpha_5 R_{(ab[c)d]} R^{(ab[c)d]}$, and $L_m(4) = \mu \bar{\Psi} \Phi \Psi - \lambda_\Phi (\Phi^+ \Phi)^2 - \lambda (\Phi^+ \Phi) \varphi^2 - \lambda_\varphi \varphi^4$.)

Interaction of Affine connection with matter fields is determined by terms containing covariant derivatives. We write these terms in a component notation, where the component labelling is done with respect to the physically important Lorenz $Spin(1, n-1)$ subgroup of $\overline{GL}(n, \mathbb{R})$. Such a labelling allows, in principle, to identify Affine field components with Lorentz fields of models based on the Poincaré symmetry. Namely, the Affine models of gravity necessarily imply existence of some symmetry breaking mechanism that reduces the global symmetry to the Poincaré one, reflecting the subgroup structure $T^n \wedge \overline{SO}(1, n-1) \subset T^n \wedge \overline{GL}(n, \mathbb{R})$. Therefore, we consider the field Ψ (and similarly for Φ field) as a sum of its Lorentz components:

$$\sum_{\substack{\{J\}\\ \{k\}\{m\}}} \Psi^{\{J\}}_{\{k\}\{m\}} |^{\{J\}}_{\{k\}\{m\}}\rangle.$$

Ket vectors in this decomposition are basis vectors of the $\{\sigma_c^\Psi\}$ representation of $\overline{SL}(n, \mathbb{R})$ group [3]. Sets of labels $\{J\}$ and $\{m\}$ determine transformation properties of a basis vector under the Lorentz $Spin(1, n-1)$ subgroup: $\{J\}$ label irreducible representation of $Spin(1, n-1)$, while numbers $\{m\}$ label particular vector within that representation. The set of parameters $\{k\}$ enumerate $Spin(1, n-1)$ multiplicity of representation $\{J\}$ within the $\{\sigma_c^\Psi\}$ representation of $\overline{SL}(n, \mathbb{R})$. These parameters $\{k\}$ are mathematically related to the left action of $Spin(n)$ subgroup in representation space $\mathcal{L}^2(Spin(n))$ of square integrable functions over the $Spin(n)$ group (for more details c.f. [3]).

The interaction term connecting fields g^{cd}, $e_d^{\ \mu}$, Γ^{ab}_μ, $\bar{\Psi}^{\{J\}}_{\{k\}\{m\}}$, $\Psi^{\{J'\}}_{\{k'\}\{m'\}}$ is now:

$$g^{cd} e_d^{\ \mu} \Gamma^{ab}_\mu \bar{\Psi}^{\{J\}}_{\{k\}\{m\}} \Psi^{\{J'\}}_{\{k'\}\{m'\}} \sum_{\substack{\{J''\}\\ \{k''\}\{m''\}}} \langle^{\{J\}}_{\{k\}\{m\}} |\gamma_c|^{\{J''\}}_{\{k''\}\{m''\}}\rangle \langle^{\{J''\}}_{\{k''\}\{m''\}} |Q_{ab}|^{\{J'\}}_{\{k'\}\{m'\}}\rangle, \tag{2}$$

while the interaction of tensorial field with connection is given by:

$$-\frac{i}{2} g^{cd} e_c^{\ \mu} e_d^{\ \nu} \Gamma^{ab}_\nu \partial_\mu \Phi^{\dagger\{J\}}_{\{k\}\{m\}} \Phi^{\{J'\}}_{\{k'\}\{m'\}} \langle^{\{J\}}_{\{k\}\{m\}} |Q_{ab}|^{\{J'\}}_{\{k'\}\{m'\}}\rangle + \tag{3}$$

$$\frac{i}{2} g^{cd} e_c^{\ \mu} e_d^{\ \nu} \Gamma^{ab}_\nu \Phi^{\dagger\{J\}}_{\{k\}\{m\}} \partial_\mu \Phi^{\{J'\}}_{\{k'\}\{m'\}} \langle^{\{J'\}}_{\{k'\}\{m'\}} |Q_{ab}|^{\{J\}}_{\{k\}\{m\}}\rangle^* + \tag{4}$$

$$\frac{1}{2} g^{cd} e_c^{\ \mu} e_d^{\ \nu} \Gamma^{ab}_\mu \Gamma^{a'b'}_\nu \Phi^{\dagger\{J\}}_{\{k\}\{m\}} \partial_\mu \Phi^{\{J'\}}_{\{k'\}\{m'\}} \cdot$$

$$\sum_{\substack{\{J''\}\\ \{k''\}\{m''\}}} \langle^{\{J\}}_{\{k\}\{m\}} |Q_{ab}|^{\{J''\}}_{\{k''\}\{m''\}}\rangle \langle^{\{J''\}}_{\{k''\}\{m''\}} |Q_{a'b'}|^{\{J'\}}_{\{k'\}\{m'\}}\rangle. \tag{5}$$

The scalar dilaton field interact only with the trace of Affine connection:

$$\tfrac{1}{2} g^{ab} e_a^{\ \mu} e_b^{\ \nu} (\partial_\mu - i\Gamma_a^{\ a}{}_\mu d_\varphi)\varphi(\partial_\nu - i\Gamma_a^{\ a}{}_\nu d_\varphi)\varphi, \tag{6}$$

where d_φ denotes dilatation charge of φ field.

In the above interaction terms we note an appearance of matrix elements of $\overline{GL}(n, \mathbb{R})$ generators, written in a basis of the Lorenz subgroup $Spin(1, n-1)$. The dilatation generator (that is, the trace Q^a_a) acts merely as multiplication by dilatation charge, so it is really the $\overline{SL}(n, \mathbb{R})$ matrix elements that should be calculated. (An infinite dimensional generalization of Dirac's gamma matrices also appear in the term (2); more on these matrices can be found in papers of Šijački [9].) However, before we illustrate how to evaluate these matrix elements, and thus how to calculate vertex coefficients, we must make some additional general remarks on $\overline{GL}(n, \mathbb{R})$ representations that correspond to physical fields.

3. Deunitarizing automorphism

We will briefly discuss the matter of unitarity of the representations corresponding to fields in Affine models. In standard, Poincaré symmetric models, gauge and matter fields have finite number of components and this fits well the experimental data. However, since the Lorenz group is a non compact one, this is made possible by the fact that the fields transform under the non-unitary representations of the Lorenz group. Note that it is only the compact $\overline{SO}(n-1)$ part of the Lorentz group that is represented unitary. If the unitary, so called Gelfand-Naimark, representations of the Lorenz group were used [10], the boosts would mix infinitely many field components, in contrary to observations.

For the same physical reasons, the Lorenz subgroup of $\overline{GL}(n, \mathbb{R})$ should act in an analogous way on $\overline{GL}(n, \mathbb{R})$ fields: boosts should be represented non unitarily and the Lorenz subgroup should reduce in finite dimensional subspaces of field components. On the other hand, much in the same way as spatial rotation part of the Lorenz group acts unitarily on Poincaré fields, it is physically favorable that the spatial "little group" $\overline{GL}(n-1, \mathbb{R})$, a subgroup of $\overline{GL}(n, \mathbb{R})$, acts unitarily on field components.

This can be elegantly accomplished by using a so called deunitarizing automorphism. Namely, there exists an inner automorphism [6], which leaves the $R_+ \otimes \overline{SL}(n-1, \mathbb{R})$ subgroup intact, and which maps the $Q_{(0k)}$, $Q_{[0k]}$ generators into $iQ_{[0k]}$, $iQ_{(0k)}$ respectively ($k = 1, 2, \ldots n-1$). Here $Q_{[ab]} = \frac{1}{2}(Q_{ab} - Q_{ba})$ denote the antisymmetric operators that generate the Lorentz subgroup $Spin(1, n-1)$, whereas $Q_{(ab)} = \frac{1}{2}(Q_{ab} + Q_{ba}) - \frac{1}{n} g_{ab} Q_c{}^c$ are the symmetric traceless operators that generate the proper n-volume-preserving deformations (shears).

The deunitarizing automorphism thus allows us to start with the unitary representations of the $\overline{SL}(n, \mathbb{R})$ subgroup, and upon its application, to identify the finite (unitary) representations of the abstract $\overline{SO}(n)$ compact subgroup with nonunitary representations of the physical Lorentz group, while the infinite (unitary) representations of the abstract $\overline{SO}(1, n-1)$ group now represent (non-unitarily) the compact $\overline{SO}(n)/\overline{SO}(n-1)$ generators.

4. Gauge Affine symmetry vertex coefficients evaluation

Now we return to evaluation of vertex coefficients for interaction between various Lorentz components of the $\overline{GL}(n,\mathbb{R})$ fields. The nontrivial part is to find matrix elements of $\overline{SL}(n,\mathbb{R})$ shear generators in expressions (2)-(5). We will do that by using formula:

$$\left\langle \begin{matrix} \{J'\} \\ \{k'\}\{m'\} \end{matrix} \middle| Q_{(ab)} \middle| \begin{matrix} \{J\} \\ \{k\}\{m\} \end{matrix} \right\rangle = \tfrac{i}{2}\sqrt{\tfrac{dim(\{J\})}{dim(\{J'\})}}\, C^{\{J\}\,\square\square\,\{J'\}}_{\{m\}\;\;ab\;\;\{m'\}}$$

$$\times \textstyle\sum_{c=2}^{n} \sqrt{\tfrac{c-1}{c}} \Big(C_2(so(c)_{\{k'\}}) - C_2(so(c)_{\{k\}}) + \tilde{\sigma}_c \Big) C^{\{J\}(\square\square)^{n-c+1}\{J'\}}_{\{k\}\;\;(0)^{c-2}\;\;\{k'\}}\,, \tag{7}$$

that was directly derived from the generalized Gell-Mann formula [3]. Capital C here denotes Clebsch-Gordan coefficients of $Spin(n)$ group and C_2 are certain second order Casimir operator values - for a detailed account of the used notation, please refer to [3].

However, formula (7) is given in the basis of the compact $Spin(n)$ subgroup, and not in the basis of the physically important Lorentz group $Spin(1,n-1)$. On the other hand, it turns out that taking into account deunitarizing automorphism exactly amounts to keeping reduced matrix element from (7) and replacing the remaining Clebsch-Gordan coefficient of the $Spin(n)$ group by the corresponding coefficient of the Lorenz group $Spin(1,n-1)$ [7].

Now, as a concrete example, we will consider tensorial Affine field Φ in $n=5$ dimensions. For all $n=5$ notation we adhere to that of our paper [2]. As a basis for $Spin(5)$ representations we pick vectors:

$$\left\{ \left| \begin{matrix} \overline{J}_1 & \overline{J}_2 \\ J_1 & J_2 \\ m_1 & m_2 \end{matrix} \right\rangle, \overline{J}_i = 0, \frac{1}{2}, \dots; \overline{J}_1 \geq \overline{J}_2; m_i = -J_i, \dots J_i \right\}. \tag{8}$$

with respect to decomposition $so(5) \supset so(4) = so(3) \oplus so(3)$. Basis of $\overline{SL}(5,\mathbb{R})$ representation space is then given by vectors [2]:

$$\left\{ \left| \begin{matrix} \overline{J}_1 & \overline{J}_2 & & \\ K_1 & K_2 & J_1 & J_2 \\ k_1 & k_2 & m_1 & m_2 \end{matrix} \right\rangle \right\}. \tag{9}$$

The reduced matrix elements of the $sl(5,\mathbb{R})$ shear (noncompact) operators, derived from an alternative form of Gell-Mann formula that we have given in the paper [2], read:

$$
\begin{aligned}
&\left\langle \begin{smallmatrix} \overline{J}'_1\,\overline{J}'_2 \\ K'_1 K'_2 \\ k'_1\; k'_2 \end{smallmatrix} \right\| Q \left\| \begin{smallmatrix} \overline{J}_1\,\overline{J}_2 \\ K_1 K_2 \\ k_1\; k_2 \end{smallmatrix} \right\rangle = \sqrt{\tfrac{dim(\overline{J}_1,\overline{J}_2)}{dim(\overline{J}'_1,\overline{J}'_2)}} \\
&\times\Bigg(\Big(\sigma_1+i\sqrt{\tfrac{4}{5}}(\overline{J}'_1(\overline{J}'_1+2)+\overline{J}'_2(\overline{J}'_2+1)-\overline{J}_1(\overline{J}_1+2)-\overline{J}_2(\overline{J}_2+1))\Big)C^{\begin{smallmatrix}\overline{J}_1\overline{J}_2 & \overline{11} & \overline{J}'_1\overline{J}'_2\end{smallmatrix}}_{\begin{smallmatrix}K_1K_2 & 00 & K'_1K'_2 \\ k_1\,k_2 & 00 & k'_1\,k'_2\end{smallmatrix}} \\
&+ i\big(\sigma_2+K'_1(K'_1+1)+K'_2(K'_2+1)-K_1(K_1+1)-K_2(K_2+1)\big)C^{\begin{smallmatrix}\overline{J}_1\overline{J}_2 & \overline{11} & \overline{J}'_1\overline{J}'_2\end{smallmatrix}}_{\begin{smallmatrix}K_1K_2 & 11 & K'_1K'_2 \\ k_1\,k_2 & 00 & k'_1\,k'_2\end{smallmatrix}} \qquad (10) \\
&- i(\delta_1+k_1-k_2)C^{\begin{smallmatrix}\overline{J}_1\overline{J}_2 & \overline{11} & \overline{J}'_1\overline{J}'_2\end{smallmatrix}}_{\begin{smallmatrix}K_1K_2 & 11 & K'_1K'_2 \\ k_1\,k_2 & 1-1 & k'_1\,k'_2\end{smallmatrix}} - i(\delta_1-k_1+k_2)C^{\begin{smallmatrix}\overline{J}_1\overline{J}_2 & \overline{1}\;\overline{1} & \overline{J}'_1\overline{J}'_2\end{smallmatrix}}_{\begin{smallmatrix}K_1K_2 & 1\;1 & K'_1K'_2 \\ k_1\,k_2 & -11 & k'_1\,k'_2\end{smallmatrix}} \\
&+ i(\delta_2+k_1+k_2)C^{\begin{smallmatrix}\overline{J}_1\overline{J}_2 & \overline{11} & \overline{J}'_1\overline{J}'_2\end{smallmatrix}}_{\begin{smallmatrix}K_1K_2 & 11 & K'_1K'_2 \\ k_1\,k_2 & 11 & k'_1\,k'_2\end{smallmatrix}} + i(\delta_2-k_1-k_2)C^{\begin{smallmatrix}\overline{J}_1\overline{J}_2 & \overline{1}\;\overline{1} & \overline{J}'_1\overline{J}'_2\end{smallmatrix}}_{\begin{smallmatrix}K_1K_2 & 1\;1 & K'_1K'_2 \\ k_1\,k_2 & -1-1 & k'_1\,k'_2\end{smallmatrix}}\Bigg),
\end{aligned}
$$

where $dim(\overline{J}_1,\overline{J}_2) = (2\overline{J}_1 - 2\overline{J}_2 + 1)(2\overline{J}_1 + 2\overline{J}_2 + 3)(2\overline{J}_1 + 2)(2\overline{J}_2 + 1)/6$ is the dimension of the $so(5)$ irreducible representation characterized by $(\overline{J}_1,\overline{J}_2)$. In this notation, $\overline{SL}(5,\mathbb{R})$ irreducible representations are labelled by parameters $\sigma_1,\sigma_2,\delta_1$ and δ_2, that appear in the formula (10).

For example, let the field Φ correspond to an unitary multiplicity free $\overline{SL}(5,\mathbb{R})$ representation, defined by labels $\sigma_2 = -4, \delta_1 = \delta_2 = 0$, with σ_1 arbitrary real. The representation space is spanned by vectors (9) satisfying $\overline{J}_1 = \overline{J}_2 = \overline{J} \in \mathbb{N}_0 + \frac{1}{2}; K_1 = K_2 = 0; J_1 = J_2 = J \leq \overline{J}$. This is a simplest class of multiplicity free representations that is unitary assuming usual scalar product. If we denote $\Phi^a, a = 1\ldots 5$ the five Φ components with $\overline{J}_1 = \overline{J}_2 = \frac{1}{2}$ (in this sense Φ^a corresponds to a Lorenz 5-vector) then the interaction vertex (3) connecting fields $\Phi^{a\dagger}$, $\partial_\mu\Phi^d$ and Affine shear connection Γ_ν^{bc} is:

$$
\frac{i}{2}\, g^{ef} e_e^{\;\mu} e_f^{\;\nu} \Phi^{a\dagger}\Gamma_\nu^{bc}\partial_\mu\Phi^d \frac{\sqrt{5}}{14}\, \sigma_1(\eta_{ab}\eta_{dc} + \eta_{ac}\eta_{db} - \frac{2}{n}\eta_{ad}\eta_{bc}). \qquad (11)
$$

To obtain this result we used an easily derivable formula for Clebsch-Gordan coefficient connecting Lorentz vector and symmetric second order Lorenz tensor representations:

$$
C^{L\square\,\square\square\,\square}_{\;\;a\;\;(bc)\;d} = \sqrt{\tfrac{n}{2(n+2)(n-1)}}(\eta_{ab}\eta_{dc} + \eta_{ac}\eta_{db} - \frac{2}{n}\eta_{ad}\eta_{bc}), \qquad (12)
$$

where we labelled $Spin(1, n-1)$ irreducible representations by Young diagrams, as in [3]. More importantly, we also used value of the reduced matrix element:

$$
\left\langle \begin{smallmatrix} \overline{\frac{1}{2}\frac{1}{2}} \\ 0\,0 \\ 0\,0 \end{smallmatrix} \right\| Q \left\| \begin{smallmatrix} \overline{\frac{1}{2}\frac{1}{2}} \\ 0\,0 \\ 0\,0 \end{smallmatrix} \right\rangle = \sqrt{\frac{2}{7}}\; \sigma_1, \qquad (13)
$$

that we obtained by using formula (10) (based on this formula, a Mathematica program was generated that directly calculates $sl(5,\mathbb{R})$ matrix elements [7], taking into account $Spin(5)$ Clebsch-Gordan coefficients found in [11]).

It is no more difficult to obtain coefficients of the vertices of the form (5). Lagrangian term (5) connecting Lorenz 5-vector Φ components Φ_5, $\Phi_5^\dagger$ and Affine connection component $\Gamma_{(55)\mu}$ is:

$$\frac{1}{15}\left(\sigma_1^2 - 25\right) g^{cd} {e_c}^{\mu} {e_d}^{\nu} \Gamma^{55}_{\mu} \Gamma^{55}_{\nu} \Phi_5^\dagger \partial_\mu \Phi_5. \tag{14}$$

Next we will consider an example where Φ field corresponds to a representation with multiplicity. Let us, again, consider 5-vector component $\overline{J}_1 = \overline{J}_2 = \frac{1}{2}$ of Φ, only this time without any restriction to the values of $\sigma_1, \sigma_2, \delta_1, \delta_2$. In general, this will correspond to a representation with non trivial multiplicity. Quantum numbers $\{k\} = (K_1, K_2, k_1, k_2)$, that label multiplicity, now can take values : $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ and $(0, 0, 0, 0)$. Therefore, this *a priori* corresponds to 5 observable 5-vector fields, differentiated by the $\{k\}$ values, and these five vector fields mutually interact by gravitational interaction. Part of the Lagrangian term (3), responsible for this interaction, has the form:

$$\frac{i}{2} g^{ef} {e_e}^{\mu} {e_f}^{\nu} \Phi^{a\dagger}_{\{k'\}} \Gamma^{bc}_{\nu} \partial_\mu \Phi^{d}_{\{k\}} \left\langle \begin{smallmatrix} \overline{\frac{1}{2}} & \overline{\frac{1}{2}} \\ K'_1 & K'_2 \\ k'_1 & k'_2 \end{smallmatrix} \middle\| Q \middle\| \begin{smallmatrix} \overline{\frac{1}{2}} & \overline{\frac{1}{2}} \\ K_1 & K_2 \\ k_1 & k_2 \end{smallmatrix} \right\rangle \frac{\sqrt{5}}{\sqrt{56}} \left(\eta_{ab}\eta_{dc} + \eta_{ac}\eta_{db} - \frac{2}{5}\eta_{ad}\eta_{bc}\right). \tag{15}$$

The reduced matrix element is obtained from the generalized Gell-Mann formula:

$$\left\langle \begin{smallmatrix} \overline{\frac{1}{2}} & \overline{\frac{1}{2}} \\ \frac{1}{2} & \frac{1}{2} \\ k'_1 & k'_2 \end{smallmatrix} \middle\| Q \middle\| \begin{smallmatrix} \overline{\frac{1}{2}} & \overline{\frac{1}{2}} \\ \frac{1}{2} & \frac{1}{2} \\ k_1 & k_2 \end{smallmatrix} \right\rangle =$$

$$\begin{aligned} \frac{1}{4\sqrt{14}} \Bigg(& - 2\sigma_1 C_3{}^{\frac{1}{2}\,0\,\frac{1}{2}}_{k_1\,0\,k'_1} C_3{}^{\frac{1}{2}\,0\,\frac{1}{2}}_{k_2\,0\,k'_2} + 15\sigma_2 C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_1\,0\,k'_1} C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,0\,k'_2} - \\ & - 15 C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_1\,-1\,k'_1} \Big((k_1 + k_2 - \delta_2) C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,-1\,k'_2} + (-k_1 + k_2 + \delta_1) C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,1\,k'_2} \Big) \\ & - 15 C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_1\,1\,k'_1} \Big((k_1 - k_2 + \delta_1) C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,-1\,k'_2} - (k_1 + k_2 + \delta_2) C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,1\,k'_2} \Big) \Bigg), \end{aligned}$$

$$\left\langle \begin{smallmatrix} \overline{\frac{1}{2}\,\frac{1}{2}} \\ 0\,0 \\ 0\,0 \end{smallmatrix} \middle\| Q \middle\| \begin{smallmatrix} \overline{\frac{1}{2}} & \overline{\frac{1}{2}} \\ K_1 & K_2 \\ k_1 & k_2 \end{smallmatrix} \right\rangle = 0, \qquad \left\langle \begin{smallmatrix} \overline{\frac{1}{2}\,\frac{1}{2}} \\ 0\,0 \\ 0\,0 \end{smallmatrix} \middle\| Q \middle\| \begin{smallmatrix} \overline{\frac{1}{2}\,\frac{1}{2}} \\ 0\,0 \\ 0\,0 \end{smallmatrix} \right\rangle = \sqrt{\frac{2}{7}}\,\sigma_1, \tag{16}$$

where C_3 denotes an usual *Spin*(3) Clebsch-Gordan coefficient.

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