3-algebras and $(2,0)$ Supersymmetry

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Abstract

We review recent work which constructs a non-Abelian system of equations with $(2,0)$ supersymmetry [1]. The result requires a totally anti-symmetric 3-algebra but the on-shell conditions imply that all the non-Abelian dynamics are restricted to five-dimensions. Some applications to M5-branes are discussed.

1. Introduction and Motivation

String Theory offers a powerful and compelling framework to discuss gravity and gauge interactions in a unified and quantum manner. However String Theory, as generally understood, is only really defined as a set of perturbative 'Feynman' rules. As such there is no nonperturbative definition and rather there exists 5 different sets of such 'rules'.

Over the past 15 years a crucial ingredient of String Theory are Dp-branes [2]. These are extended objects with p spatial dimensions which are the allowed end points of open strings. As such their quantum dynamics determined by quantizing these open strings and this leads to $(p+1)$-dimensional Yang-Mills Gauge theory and in particular a non-Abelian structure on parallel D-branes.

However there is strong - perhaps overwhelming - evidence for a single complete unifying theory known as M-theory. One 'definition' of M-Theory is as the strong coupling limit of type IIA. In this case the radius of the extra dimension is given by $R_{11} = g_s l_s$, where g_s is the string coupling constant and $l_s = \sqrt{\alpha'}$ is the string length. It is then claimed that the weak curvature effective action is 11D supergravity [3], which is uniquely determined by supersymmetry. In any case, to date there is no satisfactory microscopic description or definition. i M-theory does not contain any

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strings (otherwise we'd have a string perturbation expansion). Rather it contains 2-branes (the M2-brane) and 5-branes (the M5-brane). These can be thought of as the lifts of the D2-brane and D4-brane of type IIA string theory respectively. Since these theories are described by maximally supersymmetric Yang-Mills gauge theories we see that, at least at a formal level, the M-theory/type IIA duality implies the following: M2-branes arise as the strongly coupled limit of D2-branes. So their worldvolume theory is the IR conformal fixed point of 3D Super-Yang-Mills. M5-branes arise as the strongly coupled limit of D4-branes. Thus their worldvolume theory arises as some kind of UV completion of 5D Super-Yang-Mills.

The past few years has seen a great deal of progress in our understanding of M2-branes and in particular a description in terms of Lagrangian field theories. Technically the break through is based on the discovery of novel Chern-Simons-Matter CFT's in 3D with large amounts of supersymmetry ($N = 8, 6, ...$) ([4],[5], [6],...). The result is a fairly complete Lagrangian description of multiple M2-branes in flat space or various orbifolds of flat space.

One novel feature of these theories is that the amount of supersymmetry is determined by the gauge group, e.g.: $N = 8$ supersymmetry restricts the gauge group to $SU(2)\times SU(2)$ whereas $N = 6$ supersymmetry allows $U(n)\times U(m)$, $Sp(n) \times U(1)$. Other theories can be found with lessor amounts of supersymmetry and more general gauge groups.

The Lagrangians for M2-branes are completely specified by a 3-algebra. For our purposes a vector space V with basis T^a and a linear triple product

$$[T^A, T^B, T^C] = f^{ABC}{}_D T^D$$

and the fields take values in V, so we can expand $X^I = X^I_A T^A$, $\Psi = \Psi_A T^A$. The 3-algebra generates a Lie-algebra action on the fields X^I:

$$X^I \to \Lambda_{AB}[X^I, T^A, T^B]$$

provided that the triple product satisfies a quadratic 'fundamental' identity (generalization of Jacobi). The details of the identity vary slightly by are always equivalent to the statement that this action acts as a derivation. It turns out the the symmetries of $[T^A, T^B, T^C]$ determine the amount of supersymmetry and also the gauge group. For example in the cases mentioned above we find that for $N = 8$ supersymmetry $[T^A, T^B, T^C]$ is totally anti-symmetric whereas for $N = 6$ supersymmetry $[T^A, T^B; T_C] = -[T^B, T^A; T_C]$ and complex anti-linear in T_C.

In addition to these M2-branes M-Theory also possesses M5-branes. These are half-BPS states that have 5 extended spatial dimensions. A set of n parallel M5-branes lead to a strongly coupled 6D CFT. Very little is known about such a theory and it seems much, much harder than M2-branes (see below). Indeed hardly anything is known of conformal field theories in more than 4 dimensions. There has been much recent attention on the reduction of the M5-brane theory to four-dimensional gauge theories with and without a Lagrangian description [7].

In this talk we will review a construction of classical 6D theories with $(2,0)$ supersymmetry [1]. We will see that 3-algebras arise quite naturally. However we will also find that the non-Abelian dynamics is constrained to five dimensions. This later fact that suggests that a good first step is to look for a $(2,0)$ reformulation of D4-branes. From there one can then revisit the relation of D4-branes to the M5-brane [8].

The rest of this talk is organized as follows. In the next section we provide a review of M5-branes. Then in section 3 we give our construction of a system of equations of motion with $(2,0)$ supersymmetry. Finally in section 4 we will end with a brief summary of our conclusions and some comments.

2. M5-branes

The worldvolume of a parallel stack of M5-branes preserves 16 supersymmetries and $1+5$ dimensional Poincare symmetry along with an $SO(5)$ R-symmetry

$$SO(1,10) \to SO(1,5) \times SO(5) \qquad \mathbf{32} \to \mathbf{16}$$

In particular the preserved supersymmetries satisfy $\Gamma_{012345}\epsilon = \epsilon$ and this leads to $(2,0)$ supersymmetry in $D=6$ with Goldstinos zero modes

$$\Gamma_{012345}\Psi = -\Psi$$

and 5 scalars

$$X^I$$

The remaining Bosonic degrees of freedom arise from a self-dual tensor

$$H_{\mu\nu\lambda} = \frac{1}{3!}\,\varepsilon_{\mu\nu\lambda\rho\sigma\tau}H^{\rho\sigma\tau}$$

As we mentioned in the introduction, from the type IIA perspective the M5-brane arises as the strong coupling (UV) limit of D4-branes. Thus somehow in the strong coupling limit an extra spatial dimension arises, in other words there is an enhancement of the Poincare symmetry. The appearance of an extra spatial dimension is curious, and analogous to the type IIA to M-theory lift. This is in contrast to the case of D2-branes lifting to M2-branes where one finds an enhanced R-symmetry at the conformal fixed point.

Now, at weak coupling the effective theory of n D4-branes is five-dimensional maximally supersymmetric $U(n)$ Yang Mills. This theory is naively non-renormalizable. However M-theory implies that there is a UV completion given by the M5-brane and is a six-dimensional conformal field theory! Since no interacting 6D CFT is known and the 5D theory is non-renormalizable it is a case of the blind leading the blind, *ie.* no definition is available at either end (although there is a matrix theory attempt [9]).

One might ask where are the KK momentum modes of the M5-brane on S^1. To see this one notes that in type IIA an 11D KK mode appears as a D0-brane and D0-branes appear in the D4-brane as instanton soliton states. Such soliton states have a mass

$$m \propto \frac{1}{g_{YM}^2} \propto \frac{1}{R_{11}}$$

Thus the instantons of the 5D Yang-Mills theory have the interpretation as KK momentum of the 6D CFT on S^1 [10] and in particular

$$m \to 0 \qquad \text{as} \qquad R_{11} \to \infty \Longleftrightarrow g_{YM} \to \infty$$

However this identification has several odd features and puzzles. Firstly the momentum modes of the sixth dimension are not local with respect to other (charged) momentum modes. Secondly where are the KK modes in the Coloumb phase when the D4-branes are separated (since there a no 1/2 BPS instantons in this case)? Thirdly the Instanton moduli space is non-compact due to an arbitrary scale size. This leads to a continuous spectrum of particle states and this seems to contradict their interpretation as Kaluza-Klein states of a well-defined theory in six dimensions.

Finally the entropy of D4-branes scales as n^2 whereas that of M5-branes like n^3. This is a curious increase in the number of states which is often cited as a signature of the M-brane theory. On the other hand the D4-brane should already know about 6D of the form $\mathbf{R}^5 \times S^1$ and therefore it is of interest to try to understand the D4-brane from more of a six-dimensional perspective.

3. $(2,0)$ supersymmetry in $D = 6$

So let us now consider classical systems of equations that furnish a representation of $(2,0)$ supersymmetry in six-dimensions. First consider the free Abelian theory of a single M5-brane [12]. At linearized level the supersymmetry variations are

$$\delta X^I = i\bar{\epsilon}\Gamma^I\Psi \tag{1}$$

$$\delta\Psi = \Gamma^\mu\Gamma^I\partial_\mu X^I\epsilon + \frac{1}{3!}\frac{1}{2}\Gamma^{\mu\nu\lambda}H_{\mu\nu\lambda}\epsilon \tag{2}$$

$$\delta H_{\mu\nu\lambda} = 3i\bar{\epsilon}\Gamma_{[\mu\nu}\partial_{\lambda]}\Psi\ ,$$

and the equations of motion are those of free fields with $dH = 0$ (and hence $d \star H = 0$), $\partial_\mu\partial^\mu X^I$ and $\Gamma^\mu\partial_\mu\Psi = 0$. This theory has a nice reduction to the D4-brane theory by setting $\partial_5 = 0$ and

$$F_{\mu\nu} = H_{\mu\nu 5}$$

More generally, in the non-linear version, one finds H satisfies a non-linear self-duality which upon reduction gives

$$dF = 0 \qquad d\star\left(\frac{F}{\sqrt{1+F^2}}\right) = 0$$

Thus the non-linear self-duality condition gives rise to precisely the non-linear equations of motion of the Dirac-Born-Infeld Lagrangian [12].

We wish to generalize this algebra to non-Abelian fields. To do this we introduce a suitable covariant derivative

$$D_\mu X^I_A = \partial_\mu X^I_A - \tilde{A}_\mu{}^B{}_A X^I_B$$

Now one thing that we expect is that upon reduction we should find the supersymmetry transformations of Yang-Mills:

$$\begin{aligned}
\delta X^I &= i\bar{\epsilon}\Gamma^I\Psi \\
\delta\Psi &= \Gamma^\alpha\Gamma^I D_\alpha X^I\epsilon + \frac{1}{2}\Gamma^{\alpha\beta}\Gamma^5 F_{\alpha\beta}\epsilon - \frac{i}{2}[X^I, X^J]\Gamma^{IJ}\Gamma^5\epsilon \\
\delta A_\alpha &= i\bar{\epsilon}\Gamma_\alpha\Gamma_5\Psi\ ,
\end{aligned}$$

In order to do this we need a term in $\delta\Psi$ that is quadratic in X^I and which has a single Γ_μ. To enable this we simply postulate the existence of a new field C^μ_A.

Without going through the derivation that we give in [1], after starting with a suitably general ansatz we find the following supersymmetry transformations:

$$\begin{aligned}
\delta X^I_A &= i\bar{\epsilon}\Gamma^I\Psi_A \\
\delta\Psi_A &= \Gamma^\mu\Gamma^I D_\mu X^I_A\epsilon + \frac{1}{3!}\frac{1}{2}\Gamma_{\mu\nu\lambda}H^{\mu\nu\lambda}_A\epsilon - \frac{1}{2}\Gamma_\lambda\Gamma^{IJ}C^\lambda_B X^I_C X^J_D f^{CDB}{}_A\epsilon \\
\delta H_{\mu\nu\lambda\, A} &= 3i\bar{\epsilon}\Gamma_{[\mu\nu}D_{\lambda]}\Psi_A + i\bar{\epsilon}\Gamma^I\Gamma_{\mu\nu\lambda\kappa}C^\kappa_B X^I_C\Psi_D f^{CDB}{}_A \\
\delta\tilde{A}_\mu{}^B{}_A &= i\bar{\epsilon}\Gamma_{\mu\lambda}C^\lambda_C\Psi_D f^{CDB}{}_A \\
\delta C^\mu_A &= 0
\end{aligned}$$

where $f^{ABC}{}_D$ are totally anti-symmetric structure constants of the $N=8$ 3-algebra (possibly Lorentzian).

These transformation close and provide an example of $(2,0)$ supersymmetry with SO(5) R-symmetry and scale symmetry (C^μ_A has dimensions of length) provided that the following on-shell conditions hold:

$$\begin{aligned}
0 &= D^2 X^I_A - \frac{i}{2}\bar{\Psi}_C C^\nu_B\Gamma_\nu\Gamma^I\Psi_D f^{CDB}{}_A - C^\nu_B C_{\nu G} X^J_C X^J_E X^I_F f^{EFG}{}_D f^{CDB}{}_A \\
0 &= D_{[\mu}H_{\nu\lambda\rho]\,A} + \frac{1}{4}\epsilon_{\mu\nu\lambda\rho\sigma\tau}C^\sigma_B X^I_C D^\tau X^I_D f^{CDB}_A + \frac{i}{8}\epsilon_{\mu\nu\lambda\rho\sigma\tau}C^\sigma_B\bar{\Psi}_C\Gamma^\tau\Psi_D f^{CDB}{}_A
\end{aligned}$$

$$
\begin{aligned}
0 &= \Gamma^\mu D_\mu \Psi_A + X^I_C C^\nu_B \Gamma_\nu \Gamma^I \Psi_D f^{CDB}_A \\
0 &= \tilde{F}_{\mu\nu}{}^B{}_A - C^\lambda_C H_{\mu\nu\lambda\, D} f^{CDB}_A \\
0 &= D_\mu C^\nu_A = C^\mu_C C^\nu_D f^{BCD}{}_A \\
0 &= C^\rho_C D_\rho X^I_D f^{CDB}{}_A = C^\rho_C D_\rho \Psi_D f^{CDB}{}_A = C^\rho_C D_\rho H_{\mu\nu\lambda\, A} f^{CDB}{}_A \ ,
\end{aligned}
$$

The restriction on C^μ_A is quite strong and in effect C^μ_A picks out a fixed direction in space and in the 3-algebra. In particular without loss of generality one can take $C^\mu_A = g^2_{YM}\delta^\mu_5\delta^0_A$. As a consequence we see that the non-Abelian ($A \neq 0$) momentum modes parallel to C^μ must vanish. Thus we obtain a non-Abelian 5D Yang-Mills multiplet ($A \neq 0$) along with free 6D tensor multiplets ($A = 0$).

However it is interesting to note that we could also consider a null reduction, $x^\mu = (u, v, x^i)$:

$$
C^\mu_A = g^2_{YM}\delta^\mu_v\delta^0_A
$$

The resulting equations are ($f^{ab}{}_c = f^{0ab}{}_c$)

$$
\begin{aligned}
0 &= D^2 X^I_a - \frac{ig}{2}\bar{\Psi}_c \Gamma_v \Gamma^I \Psi_d f^{cd}{}_a \\
0 &= \Gamma^\mu D_\mu \Psi_a + g^2_{YM} X^I_c \Gamma_v \Gamma^I \Psi_d f^{cd}{}_a \\
0 &= D_{[\mu} H_{\nu\lambda\rho]\, a} - \frac{g^2_{YM}}{4}\epsilon_{\mu\nu\lambda\rho\tau v} X^I_c D^\tau X^I_d f^{cd}{}_a - \frac{ig^2_{YM}}{8}\epsilon_{\mu\nu\lambda\rho\tau v}\bar{\Psi}_c \Gamma^\tau \Psi_d f^{cd}{}_a \\
0 &= \tilde{F}_{\mu\nu}{}^b{}_a - g^2_{YM} H_{\mu\nu v\, d} f^{db}{}_a
\end{aligned}
$$

with $D_v = 0$. This gives a curious variation of Yang-Mills. In particular we find a system with 16 supersymmetries and an $SO(5)$ R-symmetry but unlike maximally supersymmetric Yang-Mills there is no potential for the scalars. The interpretation of this system is unclear but it is natural to suppose that they are related to an M5-brane with vanishing null momentum.

4. Conclusions and Comments

In this talk we have discussed some needs and oddities of the M5-brane theory. In particular we looked for interacting $(2,0)$ theories in six dimensions. We did indeed find such a system in terms of 3-algebras. However the non-Abelian dynamics is restricted to five-dimensions (with some non-interacting Abelian parts which are allowed to have six-dimensional dynamics).

In fact in an odd sense this is can be viewed as a triumph. As we have seen the Kaluza-Klein modes of the M5-brane are associated to instanton states in five-dimensional super-Yang-Mills. Thus the six-dimensional theory shouldn't have both momentum and non-Abelian instanton-like states.

We also obtained a null reduction that lead to a novel interacting system with 16 supersymmetries, $SO(5)$ R-symmetry and no potential. The most natural interpretation of this is that it describes M5-branes with vanishing null momentum.

Thus in summary the M5-brane is a rich and mysterious as ever. But hopefully some progress can be made towards defining the theory and exploring its properties.

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