On Adelic Modelling in p-Adic Mathematical Physics *

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Abstract

Adeles are infinite sequences which contain real and p-adic numbers for all primes p. They unify real and p-adic numbers, giving possibility to trait them simultaneously at an equal footing. Adelic analysis provides tools to extend some usual models with real (and complex) numbers by adding p-adic counterparts. We briefly review basic properties of adeles and their applications in p-adic mathematical physics. In particular, many adelic product formulas are presented.

1. Introduction

p-Adic numbers are invented by Kurt Hansel in 1897. Ideles and adeles are introduced in the 1930s by Claude Chevalley and André Weil, respectively. p-Adic numbers and adeles have many applications in mathematics, e.g. in representation theory, algebraic geometry and modern number theory. Since 1987, p-adic numbers and adeles have been used in construction of many models in modern mathematical physics (and related topics), what resulted in emergence of *p-adic mathematical physics* and its gradual developments. Here we consider some adelic tools in p-adic mathematical physics.

2. Adeles

On the field $\mathbb{Q}$ of rational numbers any non-trivial norm is equivalent either to the usual absolute value $|\cdot|_\infty$ or to a p-adic absolute value $|\cdot|_p$ (Ostrowski theorem). For a rational number $x = p^\nu \, \frac{a}{b}$, where integers a and $b \neq 0$ are not divisible by prime number p, by definition p-adic absolute value is $|x|_p = p^{-\nu}$ and $|0|_p = 0$. This p-adic norm is a non-Archimedean (ultrametric)

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one, because $|x+y|_p \leq \max\{|x|_p\,, |y|_p\}$. As completion of $\mathbb{Q}$ gives the field $\mathbb{Q}_\infty \equiv \mathbb{R}$ of real numbers with respect to the absolute value $|\cdot|_\infty$, by the same procedure one gets the field $\mathbb{Q}_p$ of p-adic numbers (for any prime number $p = 2,\, 3\,, 5\, \cdots$) using p-adic norm $|\cdot|_p$. Any number $x \in \mathbb{Q}_p$ has its unique canonical representation

$$x = p^{\nu(x)} \sum_{n=0}^{+\infty} x_n\, p^n\,, \quad \nu(x) \in \mathbb{Z}\,, \quad x_n \in \{0,\, 1,\, \cdots, \, p-1\},\; x_0 \neq 0. \quad (1)$$

Real and p-adic numbers, as completions of rational numbers, are unified in the form of adeles. An adele α is an infinite sequence

$$\alpha = (\alpha_\infty,\, \alpha_2,\, \alpha_3,\, \cdots,\, \alpha_p\,,\, \cdots)\,, \quad \alpha_\infty \in \mathbb{R}\,,\; \alpha_p \in \mathbb{Q}_p\,, \quad (2)$$

where for all but a finite set $\mathcal{P}$ of primes p one has that $\alpha_p \in \mathbb{Z}_p = \{x \in \mathbb{Q}_p\, :\, |x|_p \leq 1\}$. Elements of $\mathbb{Z}_p$ are called p-adic integers and they have $\nu(x) \geq 0$ in (1). The set $\mathbb{A}_\mathbb{Q}$ of all adeles, related to $\mathbb{Q}$, can be presented as

$$\mathbb{A}_\mathbb{Q} = \bigcup_{\mathcal{P}} A(\mathcal{P})\,, \quad A(\mathcal{P}) = \mathbb{R} \times \prod_{p\in\mathcal{P}} \mathbb{Q}_p \times \prod_{p\notin\mathcal{P}} \mathbb{Z}_p\,. \quad (3)$$

Endowed with componentwise addition and multiplication $\mathbb{A}_\mathbb{Q}$ is the adele ring.

The multiplicative group of ideles $\mathbb{A}_\mathbb{Q}^\times$ is a subset of $\mathbb{A}_\mathbb{Q}$ with elements $\eta = (\eta_\infty\,, \eta_2\,, \eta_3\,, \cdots, \eta_p\,, \cdots)$, where $\eta_\infty \in \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ and $\eta_p \in \mathbb{Q}_p^\times = \mathbb{Q}_p \setminus \{0\}$ with the restriction that for all but a finite set $\mathcal{P}$ one has that $\eta_p \in \mathbb{U}_p = \{x \in \mathbb{Q}_p\, :\, |x|_p = 1\}$. $\mathbb{U}_p$ is a multiplicative group of p-adic units. Thus the whole set of ideles, related to $\mathbb{Q}^\times = \mathbb{Q} \setminus \{0\}$, is

$$\mathbb{A}_\mathbb{Q}^\times = \bigcup_{\mathcal{P}} A^\times(\mathcal{P}), \quad A^\times(\mathcal{P}) = \mathbb{R}^\times \times \prod_{p\in\mathcal{P}} \mathbb{Q}_p^\times \times \prod_{p\notin\mathcal{P}} \mathbb{U}_p\,. \quad (4)$$

A principal adele (idele) is a sequence $(x, x, \cdots, x, \cdots) \in \mathbb{A}_\mathbb{Q}$, where $x \in \mathbb{Q}$ $\;(x \in \mathbb{Q}^\times)$. $\mathbb{Q}$ and $\mathbb{Q}^\times$ are naturally embedded in $\mathbb{A}_\mathbb{Q}$ and $\mathbb{A}_\mathbb{Q}^\times$, respectively. By concept of the principal adeles one straightforwardly generalizes rational numbers in such way that one takes into account all their nontrivial norms. Adeles are such generalization of principal adeles that it provides possibility to have some well-defined mathematical structures.

Let $\mathbb{P}$ be set of all primes p. Denote by $\mathcal{P}_i$, $i \in \mathbb{N}$, subsets of $\mathbb{P}$. Let us introduce an ordering by $\mathcal{P}_i \prec \mathcal{P}_j$ if $\mathcal{P}_i \subset \mathcal{P}_j$. It is evident that $A(\mathcal{P}_i) \subset A(\mathcal{P}_j)$ when $\mathcal{P}_i \prec \mathcal{P}_j$. Adelic topology in $\mathbb{A}_\mathbb{Q}$ is introduced by inductive limit: $\mathbb{A}_\mathbb{Q} = \lim \mathrm{ind}_\mathcal{P} A(\mathcal{P})$. A basis of adelic topology is a collection of open sets of the form $V(\mathcal{P}) = \mathbb{V}_\infty \times \prod_{p\in\mathcal{P}} \mathbb{V}_p \times \prod_{p\notin\mathcal{P}} \mathbb{Z}_p$, where $\mathbb{V}_\infty$ and $\mathbb{V}_p$ are open sets in $\mathbb{R}$ and $\mathbb{Q}_p$, respectively. A sequence of adeles $\alpha^{(n)} \in \mathbb{A}_\mathbb{Q}$

converges to an adele $\alpha \in \mathbb{A}_{\mathbb{Q}}$ if (i) it converges to α componentwise and (ii) if there exist a positive integer N and a set $\mathcal{P}$ such that $\alpha^{(n)}, \alpha \in A(\mathcal{P})$ when $n \geq N$. In the analogous way, these assertions hold also for idelic spaces $A^{\times}(\mathcal{P})$ and $\mathbb{A}_{\mathbb{Q}}^{\times}$. $\mathbb{A}_{\mathbb{Q}}$ and $\mathbb{A}_{\mathbb{Q}}^{\times}$ are locally compact topological spaces.

There are adelic-valued and complex-valued functions of adelic arguments. For various mathematical aspects of adeles and their functions one can use books [1, 2, 3].

3. p-Adic and adelic models

The field $\mathbb{Q}$ of rational numbers is dense not only in $\mathbb{R}$ but also in $\mathbb{Q}_p$. Some algebraic equations have solutions in $\mathbb{Q}$ if and only if they have solutions in $\mathbb{R}$ and all $\mathbb{Q}_p$ (Hasse local-global principle). This gives rise to successful application of adeles in modern number theory and algebraic geometry.

What about application of p-adic numbers and adeles in physics? Recall that measuring of physical quantities is practically related to measurement of distances and it is in agreement with the Archimedean axiom. As a result of measurements one obtains some rational numbers with distance between them induced by usual absolute value $|\cdot|_\infty$. There are no p-adic numbers as result of measurements. Hence, the corresponding mathematical models have been mainly considered using real and complex numbers. However, p-adic numbers and adeles may play very important role in deeper understanding of physical phenomena, as well as in appropriate description of some sectors of the life science.

The first significant employment of p-adic numbers in mathematical physics started in 1987 by successful construction of p-adic string amplitudes, which have p-adic valued world-sheet and real-valued momenta. Thus in p-adic systems there is something which describes by p-adic numbers and some properties which can be measured and expressed by real rational numbers. Since 1987, many p-adic models have been constructed as the corresponding counterparts of the real models. Such real and p-adic models start with the same form and can be treated simultaneously by adelic tools. For an early review of p-adic and adelic models we refer to [4, 5].

Especially adelic products have attracted much attention. They are of the form

$$\phi_\infty(x_1, \cdots, x_n; a_1, \cdots, a_m) \prod_{p \in \mathbb{P}} \phi_p(x_1, \cdots, x_n; a_1, \cdots, a_m) = C, \qquad (5)$$

where $x_i \in \mathbb{Q}$, $\quad a_j \in \mathbb{C}$, $\quad \phi_\infty$ and ϕ_p are real or complex valued functions, and C is a constant (often $C = 1$). It is obvious that expressions of the form (5) connect real and p-adic characteristics of the same object at the equal footing. Moreover, the real quantity $\phi_\infty(x_1, \cdots, x_n; a_1, \cdots, a_m)$ can be expressed as product of all p-adic inverses. This can be of practical importance when functions ϕ_p are simpler than ϕ_∞, but may also lead to more profound understanding of physical reality.

To illustrate formula (5) let us first present two very simple examples:

$$|x|_\infty \times \prod_{p\in\mathbb{P}} |x|_p = 1\,, \ \text{if } x \in \mathbb{Q}^\times\,, \quad \text{and} \quad \chi_\infty(x) \times \prod_{p\in\mathbb{P}} \chi_p(x) = 1\,, \ \text{if } x \in \mathbb{Q}\,, \tag{6}$$

where $\chi_\infty(x) = \exp(-2\pi i x)$ and $\chi_p(x) = \exp 2\pi i \{x\}_p$ are real and p-adic additive characters, respectively, and $\{x\}_p$ denotes the fractional part of x. It follows from (6) that $d_\infty(x,y) = \prod_{p\in\mathbb{P}} d_p^{-1}(x,y)$, where $d_\infty(x,y) = |x-y|_\infty$ and $d_p(x,y) = |x-y|_p$, i.e. the usual distance between any two rational points can be regarded through product of the inverse p-adic ones. One can also write $\chi_\infty(ax+bt) = \prod_{p\in\mathbb{P}} \chi_p[-(ax+bt)]$ when $a\,, b\,, x\,, t \in \mathbb{Q}$, and consider a real plane wave as composed of p-adic plane waves.

Let us also notice some adelic products related to number theory:

$$\lambda_\infty(x) \prod_{p\in\mathbb{P}} \lambda_p(x) = 1\,, \qquad \Big(\frac{x,y}{\infty}\Big) \prod_{p\in\mathbb{P}} \Big(\frac{x,y}{p}\Big) = 1\,, \tag{7}$$

where x is presented by (1) and

$$\lambda_p(x) = \begin{cases} 1, & \nu(x) = 2k\,, \quad p \neq 2\,, \\ \sqrt{\left(\frac{-1}{p}\right)}\left(\frac{x_0}{p}\right), & \nu(x) = 2k+1\,, \quad p \neq 2\,, \\ \exp\left[\pi i (x_1 + 1/4)\right], & \nu(x) = 2k\,, \quad p = 2\,, \\ \exp\left[\pi i (x_2 + x_1/2 + 1/4)\right], & \nu(x) = 2k+1\,, \quad p = 2\,, \end{cases} \tag{8}$$

$$\lambda_\infty(x) = \exp\Big(-\frac{\pi i}{4}\,\mathrm{sgn}\, x\Big), \quad \Big(\frac{x,y}{\infty}\Big) = \begin{cases} -1, & x < 0,\ y < 0\,, \\ \ \ 1, & \text{otherwise}\,, \end{cases} \tag{9}$$

$\left(\frac{x}{p}\right)$ and $\left(\frac{x,y}{p}\right)$ are Legendre and Hilbert symbols [5], respectively.

Gauss integrals satisfy adelic product formula [6]

$$\int_{\mathbb{R}} \chi_\infty(a\,x^2 + b\,x)\, d_\infty x \prod_{p\in\mathbb{P}} \int_{\mathbb{Q}_p} \chi_p(a\,x^2 + b\,x)\, d_p x = 1\,, \quad a \in \mathbb{Q}^\times\,, \ \ b \in \mathbb{Q}\,, \tag{10}$$

what follows from

$$\int_{\mathbb{Q}_v} \chi_v(a\,x^2 + b\,x)\, d_v x = \lambda_v(a)\, |2\,a|_v^{-\frac{1}{2}}\, \chi_v\Big(-\frac{b^2}{4a}\Big), \quad v = \infty\,, 2\,, \cdots, p \cdots . \tag{11}$$

These Gauss integrals apply in evaluation of the Feynman path integrals

$$\mathcal{K}_v(x'',t''; x',t') = \int_{x',t'}^{x'',t''} \chi_v\Big(-\frac{1}{h}\int_{t'}^{t''} L(\dot q, q, t)\, dt\Big)\, \mathcal{D}_v q\,, \tag{12}$$

for kernels $\mathcal{K}_v(x'',t'';x',t')$ of the evolution operator in adelic quantum mechanics [7] for quadratic Lagrangians. In the case of Lagrangian $L(\dot q,q) = \frac{1}{2}\Big(-\frac{\dot q^2}{4} - \lambda\, q + 1\Big)$ for the de Sitter cosmological model (what is similar to a particle with constant acceleration λ) one obtains [8, 9]

$$\mathcal{K}_\infty(x'',T;x',0)\prod_{p\in\mathbb{P}}\mathcal{K}_p(x'',T;x',0) = 1\,, \quad x'',x',\lambda\in\mathbb{Q}\,,\ \ T\in\mathbb{Q}^\times\,, \tag{13}$$

where

$$\mathcal{K}_v(x'',T;x',0) = \lambda_v(-8T)\,|4T|_v^{-\frac{1}{2}}\,\chi_v\Big(-\frac{\lambda^2\,T^3}{24}+[\lambda\,(x''+x')-2]\frac{T}{4}+\frac{(x''-x')^2}{8T}\Big)\,. \tag{14}$$

The adelic wave function for the simplest ground state has the form

$$\psi_{\mathbb{A}}(x) = \psi_\infty(x)\prod_{p\in\mathbb{P}}\Omega(|x|_p) = \begin{cases} \psi_\infty(x), & x\in\mathbb{Z},\\ 0, & x\in\mathbb{Q}\setminus\mathbb{Z}\,, \end{cases} \tag{15}$$

where $\Omega(|x|_p) = 1$ if $|x|_p \le 1$ and $\Omega(|x|_p) = 0$ if $|x|_p > 1$. Since this wave function is non-zero only in integer points it can be interpreted as discreteness of the space due to p-adic effects in adelic approach.

The Gel'fand-Graev-Tate gamma and beta functions [4, 5] are:

$$\Gamma_\infty(a) = \int_{\mathbb{R}} |x|_\infty^{a-1}\chi_\infty(x)\,d_\infty x = \frac{\zeta(1-a)}{\zeta(a)}\,, \tag{16a}$$

$$\Gamma_p\,(a) = \int_{\mathbb{Q}_p} |x|_p^{a-1}\chi_p(x)\,d_p x = \frac{1-|p|_p^{1-a}}{1-|p|_p^{a}}\,, \tag{16b}$$

$$B_\infty(a,b) = \int_{\mathbb{R}} |x|_\infty^{a-1}\,|1-x|_\infty^{b-1}\,d_\infty x = \Gamma_\infty(a)\,\Gamma_\infty(b)\,\Gamma_\infty(c)\,, \tag{17}$$

$$B_p(a,b) = \int_{\mathbb{Q}_p} |x|_p^{a-1}\,|1-x|_p^{b-1}\,d_p x = \Gamma_p(a)\,\Gamma_p(b)\,\Gamma_p(c)\,, \tag{18}$$

where $a,b,c\in\mathbb{C}$ with condition $a+b+c=1$ and $\zeta(a)$ is the Riemann zeta function. With a regularization of the product of p-adic gamma functions one has adelic products:

$$\Gamma_\infty(u)\prod_{p\in\mathbb{P}}\Gamma_p(u) = 1\,, \quad B_\infty(a,b)\prod_{p\in\mathbb{P}}B_p(a,b) = 1\,, \tag{19}$$

where $u\neq 0,1\,,\quad u = a,b,c\,,\quad a+b+c=1$. It is worth noting now that $B_\infty(a,b)$ and $B_p(a,b)$ are the crossing symmetric standard and p-adic Veneziano amplitudes for scattering of two open tachyon strings. There are generalizations of the above product formulas to integration on quadratic extensions of $\mathbb{R}$ and $\mathbb{Q}_p$, as well as on algebraic number fields, and they include scattering of closed strings [5, 10].

Introducing real, p-adic and adelic zeta functions as

$$\zeta_\infty(a) = \int_{\mathbb{R}} \exp\left(-\pi\, x^2\right) |x|_\infty^{a-1}\, d_\infty x = \pi^{-\frac{a}{2}}\, \Gamma\Big(\frac{a}{2}\Big)\,, \tag{20}$$

$$\zeta_p(a) = \frac{1}{1-|p|_p} \int_{\mathbb{Q}_p} \Omega(|x|_p)\, |x|_p^{a-1}\, d_p x = \frac{1}{1-|p|_p^a}\,, \quad \mathrm{Re}\, a > 1\,, \tag{21}$$

$$\zeta_{\mathbb{A}}(a) = \zeta_\infty(a) \prod_{p\in\mathbb{P}} \zeta_p(a) = \zeta_\infty(a)\zeta(a)\,, \tag{22}$$

and

$$\zeta_{\mathbb{A}}(1-a) = \zeta_{\mathbb{A}}(a)\,, \tag{23}$$

where $\zeta_{\mathbb{A}}(a)$ can be called adelic zeta function, from (23) one obtains functional equation for the Riemann zeta function. Let us note that $\exp\left(-\pi\, x^2\right)$ and $\Omega(|x|_p)$ are analogous functions in real and p-adic cases. Adelic harmonic oscillator [7] has connection with the Riemann zeta function. Namely, the simplest vacuum state of the adelic harmonic oscillator is the following Schwartz-Bruhat function:

$$\psi_{\mathbb{A}}(x) = 2^{\frac{1}{4}}\, e^{-\pi\, x_\infty^2} \prod_{p\in\mathbb{P}} \Omega(|x_p|_p)\,, \tag{24}$$

whose the Fourier transform

$$\psi_{\mathbb{A}}(k) = \int \chi_{\mathbb{A}}(k\, x)\, \psi_{\mathbb{A}}(x) = 2^{\frac{1}{4}}\, e^{-\pi\, k_\infty^2} \prod_{p\in\mathbb{P}} \Omega(|k_p|_p) \tag{25}$$

has the same form as $\psi_{\mathbb{A}}(x)$. The Mellin transform of $\psi_{\mathbb{A}}(x)$ is

$$\Phi_{\mathbb{A}}(a) = \int \psi_{\mathbb{A}}(x)\, |x|^a\, d_{\mathbb{A}}^{\times} x =$$

$$\int_{\mathbb{R}} \psi_\infty(x)\, |x|^{a-1} d_\infty x \prod_{p\in\mathbb{P}} \frac{1}{1-|p|_p} \int_{\mathbb{Q}_p} \Omega(|x|_p)|x|^{a-1}\, d_p x = \sqrt{2}\, \Gamma\Big(\frac{a}{2}\Big)\, \pi^{-\frac{a}{2}}\, \zeta(a) \tag{26}$$

and the same for $\psi_{\mathbb{A}}(k)$. Then according to the Tate formula one obtains (23). It is remarkable that such simple physical system as harmonic oscillator is related to so significant mathematical object as the Riemann zeta function.

Adelic properties of dynamical systems, which evolution is governed by linear fractional transformations [11]

$$f(x) = \frac{ax+b}{cx+d}\,, \quad a, b, c, d, \in \mathbb{Q}\,, \quad ad - bc = 1 \tag{27}$$

is also investigated. It is shown that rational fixed points are p-adic indifferent for all but a finite set $\mathcal{P}$ of primes, i.e. only for finite number of p-adic cases a rational fixed point may be attractive or repelling.

Recently [12] wavelet analysis was considered on adeles.

4. Concluding remarks

We presented a brief review of some basic adelic tools in p-adic mathematical physics. We considered above simple cases of adeles $\mathbb{A}_{\mathbb{Q}}$ consisting of completions of $\mathbb{Q}$. There is also ring of adeles $\mathbb{A}_{\mathbb{K}}$ related to the completions of any global field $\mathbb{K}$. There is a straightforward generalization of $\mathbb{A}_{\mathbb{Q}}$ to the n-dimensional vector space $\mathbb{A}_{\mathbb{Q}}^n = \prod_{i=1}^n \mathbb{A}_{\mathbb{Q}}^{(i)}$ (see, e.g. [1]). Adelic algebraic group $G(\mathbb{A}_{\mathbb{K}})$ is an adelization of a linear algebraic group G over completion fields $\mathbb{K}_v$ of a global field $\mathbb{K}$ [1, 2, 3].

For a more detail insight into this attractive and promising field of investigations let us also mention a few additional topics. Adelic quantum cosmology (for a review, see [9]) is an application of adelic quantum mechanics [7] to explore very early evolution of the universe as a whole. Adelic path integral [13] is a suitable extension of the standard Feynman path integral and serves to describe quantum evolution of adelic objects. Conjecture on the adelic universe with real and p-adic worlds, as well as p-adic origin of dark matter and dark energy are discussed in [9].

Adelic summability [14] of perturbation series is an approach to summation of divergent series in the real case when they are convergent in all p-adic cases. In a few papers [14], rational sums are obtained for many p-adic series with factorials.

Use of effective Lagrangians on real numbers for p-adic strings has been very efficient in their application to string theory and cosmology. Paper [15] is an attempt towards effective Lagrangian for adelic strings without tachyons. Further development of adelic analysis and, in particular, adelic generalized functions [6, 16, 17] is one of mathematical opportunities. Let us also mention work towards adelic superanalysis [18].

One can conclude that there has been a successful application of adeles in p-adic mathematical physics and that one can expect a growing interest in their further mathematical developments as well as in applications. Recent review of p-adic mathematical physics is presented in [19].

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