Curved Dp-brane in curved background by canonical methods*

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Abstract

We give a comparison of the non-commutativity parameters for the space-time coordinates in the theories of the bosonic string moving in the constant and weakly curved background. We explain and resolve the discrepancies that exist in the literature regarding the appearance of the momenta dependent terms.

1. Introduction

We will investigate the non-commutativity [1]-[6] of the open string coordinates in the background which is the solution of the space-time equations of motion. Working with general solution for the metric tensor $G_{\mu\nu}(x)$ and Kalb-Ramond field $B_{\mu\nu}(x)$ would be very complicated. In the great majority of papers the simplest solution, the flat background, $G_{\mu\nu} = const$ and $B_{\mu\nu} = const$, is used. A particular solution of the space-time equation of motion is weakly curved background [2] with $G_{\mu\nu} = const$ and linearly dependent Kalb-Ramond field $B_{\mu\nu} = b_{\mu\nu} + \frac{1}{3}B_{\mu\nu\rho}x^\rho$, where the field strength $B_{\mu\nu\rho}$ is infinitesimally small.

Different methods, addressing the problem of weakly curved background do not give the same form of the noncommutativi- ty parameter. We resolve these discrepancies, by obtaining the complete form of the non-commutativity parameter and by explaining their roots.

Following [3, 4] we develop the canonical method in which we treat the boundary conditions (obtained from the string action principle) as constraints. By requiring the consistency of boundary conditions as constraints, we obtain the infinite set of constraints. Instead of working with

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them explicitly, we form only one σ-dependent constraint at each string endpoint. They are of the second class and their solution allows us to express the initial canonical variables in terms of the effective ones.

We investigate the non-commutativity of the space-time coordinates x^μ which is nontrivial because x^μ depends on both effective coordinates and momenta.

2. Definition of the model

The action, describing the open string propagation in the curved background [5] has a form

$$S=\kappa\int_\Sigma d^2\xi\sqrt{-g}\Big[\frac{g^{\alpha\beta}}{2}G_{\mu\nu}(x)+\frac{\epsilon^{\alpha\beta}}{\sqrt{-g}}B_{\mu\nu}(x)\Big]\partial_\alpha x^\mu\partial_\beta x^\nu, \tag{1}$$

where $x^\mu(\xi),\ \mu=0,1,...,D-1$ are the coordinates of the D-dimensional space-time, and $\xi^\alpha(\xi^0=\tau,\ \xi^1=\sigma)$ parametrize 2-dim world-sheet, with intrinsic metric $g_{\alpha\beta}(\xi)$ $(g=detg_{\alpha\beta})$.

Due to the quantum world-sheet conformal invariance, the background fields: space-time metric $G_{\mu\nu}(x)$ and the Kalb-Ramond antisymmetric field $B_{\mu\nu}(x)$ must satisfy the space-time equations of motion

$$R_{\mu\nu}-\frac{1}{4}B_{\mu\rho\sigma}B_\nu^{\ \rho\sigma}=0,\quad D_\rho B^\rho_{\ \mu\nu}=0, \tag{2}$$

where $B_{\mu\nu\rho}=\partial_\mu B_{\nu\rho}+\partial_\nu B_{\rho\mu}+\partial_\rho B_{\mu\nu}$ is the field strength of $B_{\mu\nu}$, $R_{\mu\nu}$ is the Ricci tensor and D_μ is covariant derivative. We will consider the following solution

$$G_{\mu\nu}=const,\quad B_{\mu\nu}=b_{\mu\nu}+\frac{1}{3}B_{\mu\nu\rho}x^\rho, \tag{3}$$

where the field strength of the Kalb-Ramond field $B_{\mu\nu\rho}$ is constant and infinitesimally small. We will work up to the first order in $B_{\mu\nu\rho}$ so that *weakly curved* background defined in (3) is the solution of equation (2).

From the action principle

$$\delta S=\int d\xi^2\Big[\frac{\partial\mathcal{L}}{\partial x^\mu}-\partial_\tau\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}-\partial_\sigma\frac{\partial\mathcal{L}}{\partial x'^\mu}\Big]\delta x^\mu+\int d\tau\Big[\frac{\partial\mathcal{L}}{\partial x'^\mu}\delta x^\mu\Big]\Big|_0^\pi, \tag{4}$$

we obtain the equation of motion

$$\ddot{x}^\mu=x''^\mu-2B^\mu_{\ \nu\rho}\dot{x}^\nu x'^\rho, \tag{5}$$

and boundary conditions

$$\Big[\gamma_0^\mu\delta x_\mu\Big]\Big|_{\sigma=0,\pi}=0,\quad \gamma_0^\mu=x'^\mu-2(G^{-1}B)^\mu_{\ \nu}\dot{x}^\nu. \tag{6}$$

3. The Boundary conditions as constraints

The closed string fulfills the boundary condition because $x^\mu(0) = x^\mu(\pi)$. For the open string we can impose either Neumann boundary condition, where $\Big[\delta x^\mu\Big]\Big|_0$, $\Big[\delta x^\mu\Big]\Big|_\pi$ are arbitrary i.e. string end-points can move freely, which produces

$$\gamma_\mu^0\Big|_{\sigma=0} = 0, \quad \gamma_\mu^0\Big|_{\sigma=\pi} = 0, \tag{7}$$

or the Dirichlet boundary condition

$$x^\mu\Big|_{\sigma=0} = const, \quad x^\mu\Big|_{\sigma=\pi} = const, \tag{8}$$

where the edges of the string are fixed.

We impose the Neumann boundary condition, and we treat $\gamma_0^\mu\Big|_{\sigma=0,\pi}$ as constraints. Because they must be conserved in time we obtain the secondary constraint and consecutively the infinite set of constraints

$$\gamma_\mu^n\Big|_{\sigma=0,\pi} = 0, \quad \gamma_\mu^n \equiv \dot{\gamma}_\mu^{n-1}, \quad (n \geq 1). \tag{9}$$

Their explicit form is

$$\begin{aligned}\gamma_\mu^{2n} &= \gamma_\mu^{(2n)} - \frac{2}{3} B_{\mu\alpha\beta} \sum_{k=0}^{n-1} \alpha_{2n}^k Q_k^{(2n-2k-1)\alpha\beta} \\ &\quad + 4 b_\mu{}^\nu B_{\nu\alpha\beta} \sum_{k=0}^{n-1} \alpha_{2n}^k R_k^{(2n-2k-2)\alpha\beta}, \quad (n \geq 1)\end{aligned} \tag{10}$$

$$\begin{aligned}\gamma_\mu^{2n+1} &= \tilde{\gamma}_\mu^{(2n+1)} - \frac{2}{3} B_{\mu\alpha\beta} \sum_{k=0}^{n-1} \alpha_{2n}^k R_k^{(2n-2k-1)\alpha\beta} \\ &\quad + 4 b_\mu{}^\nu B_{\nu\alpha\beta} \sum_{k=0}^{n} \alpha_{2n+2}^k Q_k^{(2n-2k)\alpha\beta}, \quad (n \geq 1)\end{aligned}$$

where we defined

$$\begin{aligned}&\gamma_\mu = G_{\mu\nu} x'^\nu - 2 B_{\mu\nu} \dot{x}^\nu, \qquad \tilde{\gamma}_\mu = G_{\mu\nu} \dot{x}^\nu - 2 B_{\mu\nu} x'^\nu \\ &Q_n^{\alpha\beta} = \dot{x}^{(n)\alpha} x^{(n+1)\beta}, \quad R_n^{\alpha\beta} = x^{(n+2)\alpha} x^{(n+1)\beta} + \dot{x}^{(n)\alpha} \dot{x}^{(n+1)\beta}.\end{aligned} \tag{11}$$

Instead of working with this infinite set of constraints, we form σ-dependent constraint at $\sigma = 0$

$$\Gamma^\mu(\sigma) \equiv \sum_{n=0}^\infty \frac{\sigma^{2n}}{(2n)!} \gamma_{2n}^\mu\Big|_{\sigma=0} + \sum_{n=0}^\infty \frac{\sigma^{2n+1}}{(2n+1)!} \gamma_{2n+1}^\mu\Big|_{\sigma=0} = \Gamma_S^\mu(\sigma) + \Gamma_A^\mu(\sigma). \tag{12}$$

Introducing Ω-even and odd variables ($\Omega : \sigma \to -\sigma$)

$$\begin{aligned} q^\mu &= \tfrac{1}{2}(1+\Omega)x^\mu, & \bar{q}^\mu &= \tfrac{1}{2}(1-\Omega)x^\mu, \\ p_\mu &= \tfrac{1}{2}(1+\Omega)\pi_\mu, & \bar{p}_\mu &= \tfrac{1}{2}(1-\Omega)\pi_\mu, \end{aligned} \tag{13}$$

and applying the method of Refs. [3, 4], we obtain the compact form of the symmetric and antisymmetric part of the constraint

$$\begin{aligned} \Gamma^S_\mu(\sigma) &= G_{\mu\nu}\bar{q}'^\nu - 2b_{\mu\nu}\dot{q}^\nu - \frac{2}{3}B_{\mu\nu\rho}\Big[\dot{q}^\nu q^\rho + \frac{1}{2}\dot{Q}^\nu q'^\rho + \frac{3}{2}\dot{\bar{q}}^\nu\bar{q}^\rho\Big] \\ &+ 2b_\mu{}^\rho B_{\rho\alpha\beta}\Big[q'^\alpha\bar{q}^\beta + \dot{Q}^\alpha\dot{\bar{q}}^\beta\Big], \\ \Gamma^A_\mu(\sigma) &= G_{\mu\nu}\dot{\bar{q}}^\nu - 2b_{\mu\nu}q'^\nu - \frac{2}{3}B_{\mu\nu\rho}\Big[q'^\nu q^\rho + \frac{1}{2}\dot{Q}^\nu\dot{q}^\rho + \frac{3}{2}\bar{q}'^\nu\bar{q}^\rho\Big] \\ &+ 2b_\mu{}^\rho B_{\rho\alpha\beta}\frac{\partial}{\partial\sigma}\Big[\dot{Q}^\alpha\bar{q}^\beta\Big], \end{aligned} \tag{14}$$

where $Q^\mu(\sigma) = \int_0^\sigma d\eta q^\mu(\eta)$.

4. From Lagrangian to Hamiltonian form of constraints

Both σ-dependent constraints (14) have a form

$$\Gamma^\mu(\sigma) = \Gamma^\mu(q, \dot{q}, q', \bar{q}, \dot{\bar{q}}, \bar{q}', \dot{Q}). \tag{15}$$

Notice that they depend not only on Ω-even and odd variables q^μ and $\bar{q}^\mu$, which would be expected at the first glance but on $\dot{Q}$, also.

We introduce canonical momenta corresponding to coordinate x^μ

$$\pi_\mu = \kappa(G_{\mu\nu}\dot{x}^\nu - 2B_{\mu\nu}x'^\nu), \tag{16}$$

and rewrite these constraints in the canonical form

$$\Gamma^\mu(\sigma) = \Gamma^\mu(q, q', p, \bar{q}, \bar{q}', \bar{p}, P), \tag{17}$$

where $P_\mu(\sigma) = \int_0^\sigma d\eta p_\mu(\eta)$. Explicitly, we have [4]

$$\begin{aligned} \Gamma^S_\mu(\sigma) &= G^E_{\mu\nu}[q]\bar{q}'^\nu - \frac{2}{\kappa}B_\mu{}^\nu[q]p_\nu - \Big[2(bh+hb) - 6h(bq) - 24bh(bq)b\Big]'_{\mu\nu}\bar{q}^\nu \\ &- \frac{1}{\kappa}\Big[h - 12bh(bq)\Big]'{}_\mu{}^\nu P_\nu - \frac{3}{\kappa}\Big[\bar{h} + 4bh(b\bar{q})\Big]_\mu{}^\nu\bar{p}_\nu + \frac{6}{\kappa^2}\Big[bh(\bar{p})\Big]_\mu{}^\nu P_\nu, \end{aligned} \tag{18}$$

and

$$\begin{aligned} \Gamma^A_\mu(\sigma) &= \frac{1}{\kappa}\bar{p}_\mu - \Big[\bar{h} - 12b\bar{h}b - 4h(b\bar{q})b + 12bh(b\bar{q})\Big]_{\mu\nu}\bar{q}'^\nu \\ &+ \frac{2}{\kappa}\Big[3b\bar{h} + h(b\bar{q})\Big]_\mu{}^\nu p_\nu + \frac{2}{\kappa}\Big[3b\bar{h} - h(b\bar{q})\Big]'{}_\mu{}^\nu P_\nu - \frac{1}{\kappa^2}h_\mu{}^\nu(p)P_\nu, \end{aligned} \tag{19}$$

where $G^E_{\mu\nu}(G,B) \equiv G_{\mu\nu} - 4B_{\mu\rho}(G^{-1})^{\rho\sigma}B_{\sigma\nu}$ is open string metric and $h_{\mu\nu}(x) \equiv \frac{1}{3}B_{\mu\nu\rho}x^\rho = \frac{1}{3}B_{\mu\nu\rho}(q+\bar{q})^\rho \equiv h^{\mu\nu} + \bar{h}^{\mu\nu}$ is infinitesimal.

The constraints are closed on themselves, because by definition

$$\dot{\Gamma}^\mu = \{H_c, \Gamma^\mu(\sigma)\} = \Gamma'^\mu(\sigma), \tag{20}$$

and so there are no more constraints. They are of the second class

$$\{\Gamma^\mu(\sigma), \Gamma^\nu(\bar{\sigma})\} = -\kappa G^E_{\mu\nu}[(q + 2b\bar{q} + \frac{1}{\kappa}P)(\sigma)]\delta'(\sigma - \bar{\sigma}) + \cdots \tag{21}$$

if the open string metric is regular, $G^E_{\mu\nu}[\star] \neq 0$.

5. The solution of the constraints and effective background

We are going to solve the second class constraints $\Gamma^\mu_S(\sigma) = 0, \quad \Gamma^\mu_A(\sigma) = 0$. We can express the Ω-odd variables in terms of the Ω-even ones $\bar{q}^\mu = f(q,p,P)$, $\bar{p}_\mu = f(p,P)$. Explicitly, in the zero order in small parameter $B_{\mu\nu\rho}$ we have

$$\bar{q}'^\mu_0 = -2\theta^{\mu\nu}_0 p_\nu \quad \to \quad \bar{q}^\mu_0 = -2\theta^{\mu\nu}_0 P_\nu, \qquad \bar{p}^0_\mu = 0, \tag{22}$$

where $\theta^{\mu\nu}_0 \equiv -\frac{1}{\kappa}(g^{-1})^{\mu\rho}b_{\rho\sigma}(G^{-1})^{\sigma\nu}$ and $g_{\mu\nu} = G^E_{\mu\nu}(G,b)$.

In the first order in $B_{\mu\nu\rho}$ the solution obtained in Ref. [4] is equal to

$$\begin{aligned}
x^\mu(\sigma) &= q^\mu(\sigma) + \bar{q}^\mu(\sigma) \\
&= q^\mu(\sigma) - \int_0^\sigma d\eta \Big[2\theta^{\mu\nu}[q(\eta)]p_\nu(\eta) + \Lambda'^{\mu\nu}_-[q(\eta)]P_\nu(\eta)\Big], \\
\pi_\mu(\sigma) &= p_\mu(\sigma) + \bar{p}_\mu(\sigma) \\
&= p_\mu + \Big[Gb^{-1}\beta(\bar{q})g^{-1}\Big]_\mu^{\ \nu} p_\nu,
\end{aligned} \tag{23}$$

where we introduced the notation

$$\begin{aligned}
\Lambda^{\mu\nu}_\pm[x] &\equiv -\frac{1}{\kappa}(G^{-1}_E)^{\mu\alpha}[B_{\alpha\beta} \pm \frac{1}{2}G_{\alpha\beta}](G^{-1})^{\beta\nu}, \\
\theta^{\mu\nu}(G,B) &\equiv -\frac{1}{\kappa}(G^{-1}_E(G,B))^{\mu\rho}B_{\rho\sigma}(G^{-1})^{\sigma\nu}, \\
\beta_{\mu\nu}(\bar{q}) &= 2\Big[b\bar{h}b - 3b^2\bar{h} - \frac{1}{4}b\bar{h}(b^{-1}\bar{q}) + 3b^2\bar{h}(b^{-1}\bar{q})b\Big]_{\mu\nu}.
\end{aligned} \tag{24}$$

We will call this case, when space-time metric is constant and Kalb-Ramond field linear in coordinate ($B_{\mu\nu\rho} \neq 0, b_{\mu\nu} \neq 0$) the case III. In the case I, when both background fields are constant ($B_{\mu\nu\rho}=0$), the solution reduces to that of Ref. [5]

$$x^\mu(\sigma)=q^\mu-2\theta_0^{\mu\nu}P_\nu,\quad \pi_\mu(\sigma)=p_\mu. \tag{25}$$

In the case II, for infinitesimal Kalb-Ramond field ($B_{\mu\nu\rho}\neq 0, b_{\mu\nu}=0$) the solution reduces to that of Ref. [3]

$$\begin{aligned}x^\mu(\sigma)&=q^\mu(\sigma)-2\int_0^\sigma d\sigma_0\Big(\theta^{\mu\nu}[q]p_\nu+\frac{1}{2}\theta'^{\mu\nu}[q]P_\nu\Big)(\sigma_0),\\ \pi_\mu(\sigma)&=p_\mu(\sigma)-\theta_\mu{}^\nu[p(\sigma)]P_\nu(\sigma).\end{aligned} \tag{26}$$

Substituting the solution (23) into the Hamiltonian we obtain the effective Hamiltonian in terms of the effective canonical variables. Comparing it with the initial Hamiltonian we find the effective background fields in the case III

$$\begin{aligned}G_{\mu\nu}&\to G^E_{\mu\nu}[q]\equiv G^{eff}_{\mu\nu}[q]\\ B_{\mu\nu}(x)&\to[\bar{h}+4b\bar{h}b]_{\mu\nu}+\beta^A_{\mu\nu}(\bar{q})\equiv B^{eff}_{\mu\nu}[\bar{q}].\end{aligned} \tag{27}$$

In both cases I and II the metric tensor remains constant $G_{\mu\nu}\to G_{\mu\nu}$ and the effective Kalb-Ramond field does not appear $B_{\mu\nu}\to 0$.

6. Non-commutativity of the closed string coordinate x^μ

Let us analyze the results (23), (25) and (26). The space-time coordinates, obviously do not commute, because they depend on both effective coordinates q^μ and momenta p_μ. Using the Poisson bracket

$$\{q^\mu(\sigma),p_\nu(\bar{\sigma})\}=\delta^\mu_\nu\delta_S(\sigma,\bar{\sigma}),\quad \delta_S(\sigma,\bar{\sigma})=\frac{1}{2}[\delta(\sigma-\bar{\sigma})+\delta(\sigma+\bar{\sigma})], \tag{28}$$

we can investigate the form of the non-commutativity parameter for these three solutions. From the solutions (25),(26) ,(23) omitting unphysical terms we obtain respectively

$$\{x^\mu(\sigma),x^\nu(\bar{\sigma})\}=2\theta_0^{\mu\nu}\theta(\sigma+\bar{\sigma}), \tag{29}$$

$$\{x^\mu(\sigma),x^\nu(\bar{\sigma})\}=\Big\{\theta^{\mu\nu}[q(\sigma)]+\theta^{\mu\nu}[q(\bar{\sigma})]\Big\}\theta(\sigma+\bar{\sigma}), \tag{30}$$

$$\begin{aligned}\{x^\mu(\sigma),x^\nu(\bar{\sigma})\}=&\Big[E^{\mu\nu}(\bar{\sigma})-E^{\nu\mu}(\sigma)\Big]\theta(\sigma+\bar{\sigma})\\ &-\Big[I^{\mu\nu}(\bar{\sigma})+I^{\nu\mu}(\sigma)\Big]\theta(\sigma-\bar{\sigma}),\end{aligned} \tag{31}$$

where we introduced

$$E^{\mu\nu}[q,\bar{q}] = \theta^{\mu\nu}[q] - I^{\mu\nu}[\bar{q}],$$
$$I^{\mu\nu}[\bar{q}] = \theta^{\mu\nu}{}_{\rho}\bar{q}^{\rho} - \theta^{\mu\nu}_{eff}[\bar{q}], \tag{32}$$
$$\theta^{\mu\nu}_{eff} = \theta^{\mu\nu}(G^{eff}, B^{eff}) = -\frac{1}{\kappa}\Big[g^{-1}B_{eff}[\bar{q}]g^{-1}\Big]^{\mu\nu},$$

and $\theta^{\mu\nu}{}_{\rho} \equiv \theta_0^{\mu\alpha}\theta_0^{\nu\beta}\theta_{\alpha\beta\rho} = \theta_0^{\mu\alpha}\theta_0^{\nu\beta}(\partial_\alpha\theta^{-1}_{\beta\rho} + \partial_\beta\theta^{-1}_{\rho\alpha} + \partial_\rho\theta^{-1}_{\alpha\beta})$.

The essential difference between two cases in the weakly curved backgrounds, is that in the case II the term $\theta^{\mu\nu}[q(\eta)]$ in the expression for x^μ is infinitesimal and in the case III it contains the finite constant part $\theta_0^{\mu\nu}$. For $b_{\mu\nu} \neq 0$ the Poisson bracket $\{\bar{q}^\mu, \bar{q}^\nu\}$ can not be neglected and beside the coordinate dependent term it produces two new terms which depend on $\bar{q}_0^\mu = -2\theta_0^{\mu\nu}P_\nu$, (where P_ν is the σ-integral of the effective momenta p_ν).

Separating center of mass variable $x^\mu(\sigma) = x^\mu_{cm} + X^\mu(\sigma)$, we obtain that in the cases I and II the non-commutativity parameter is nontrivial only on the string endpoints

$$\{X^\mu, X^\nu\} = \mp\theta_0^{\mu\nu}, \qquad (B_{\mu\nu\rho} = 0, b_{\mu\nu} \neq 0)$$
$$\{X^\mu, X^\nu\} = \mp\theta^{\mu\nu}(q), \qquad (B_{\mu\nu\rho} \neq 0, b_{\mu\nu} = 0), \tag{33}$$

for $\sigma = 0, \pi$ respectively, while the interior of the string is commutative.

The term with $\theta(\sigma - \bar{\sigma})$ in (31), which is nontrivial in the case III, produces the non-commutativity along the whole string

$$\{X^\mu(\sigma), X^\nu(\bar{\sigma})\} = \begin{cases} -\theta^{\mu\nu}[q(0)] - \frac{2}{\pi}{}_\star I^{\mu\nu}_{cm}, & \sigma = \bar{\sigma} = 0 \\ \theta^{\mu\nu}[q(\pi)] + \frac{2}{\pi}{}^\star I^{\mu\nu}_{cm}, & \sigma = \bar{\sigma} = \pi \\ (1 - \frac{2\sigma}{\pi})I^{\mu\nu}(\sigma) + \frac{2}{\pi}\Big[{}_\star I^{\mu\nu}(\sigma) - {}_\star I^{\mu\nu}_{cm}\Big], & \sigma = \bar{\sigma} \neq 0, \pi \\ \Big\{I^{\mu\nu}(\sigma) - \frac{1}{\pi}\Big[\sigma I^{\mu\nu}(\sigma) + \bar{\sigma}I^{\mu\nu}(\bar{\sigma})\Big] & \\ +\frac{1}{\pi}\Big[{}_\star I^{\mu\nu}(\sigma) + {}_\star I^{\mu\nu}(\bar{\sigma}) - 2{}_\star I^{\mu\nu}_{cm}\Big]\Big\}, & \sigma > \bar{\sigma} \\ \Big\{I^{\mu\nu}(\bar{\sigma}) - \frac{1}{\pi}\Big[\sigma I^{\mu\nu}(\sigma) + \bar{\sigma}I^{\mu\nu}(\bar{\sigma})\Big] & \\ +\frac{1}{\pi}\Big[{}_\star I^{\mu\nu}(\sigma) + {}_\star I^{\mu\nu}(\bar{\sigma}) - 2{}_\star I^{\mu\nu}_{cm}\Big]\Big\}, & \sigma < \bar{\sigma} \end{cases}, \tag{34}$$

where

$$ {}_\star I^{\mu\nu}(\sigma) = \int_0^\sigma d\eta\; I^{\mu\nu}(\eta), \quad {}^\star I^{\mu\nu}(\sigma) = \int_\sigma^\pi d\eta\; I^{\mu\nu}(\eta), $$
$$ {}_\star I^{\mu\nu}_{cm} = \frac{1}{\pi}\int_0^\pi d\eta\; {}_\star I^{\mu\nu}(\eta), \quad {}^\star I^{\mu\nu}_{cm} = \frac{1}{\pi}\int_0^\pi d\eta\; {}^\star I^{\mu\nu}(\eta). \tag{35} $$

In the case I the non-commutativity parameter is constant, in the case II it is coordinate dependent and in the case III it is coordinate and momenta dependent.

7. Discussion

We investigated different forms of non-commutativity relations between space-time coordinates. It is well known that, the non-commutativity parameter is constant for flat background $\theta^{\mu\nu} = const$. We showed that, in the case of weakly curved background for $b_{\mu\nu} = 0$ it depends on coordinates only $\theta^{\mu\nu}(q)$, and for $b_{\mu\nu} \neq 0$ on both coordinates and momenta. The momenta dependent parameter has not been observed by the path integral method [2], but one part of our result has been obtained by canonical approach [6].

We proved that the momenta dependent terms exist. Apart from the term obtained in [6], we obtained additional term $\theta^{\mu\nu}_{eff}(\bar{q})$, (32) which depends on the effective background fields in the same way as the initial non-commutativity parameter depends on the initial background fields.

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