Integrability versus chaos in non-autonomous Hamiltonian systems. Applications to the study of some transport phenomena*

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Abstract

The phase space of some Hamiltonian systems is a complex mixture of invariant zones whose points have regular, respectively chaotic dynamics. The regular zones (where the system is almost integrable) are characterized by a reduced transport. Such zones act sometimes as transport barriers which separate different chaotic zones. Inside the chaotic zone the transport is larger, due to the mixing properties of the system. In this paper we propose general results concerning the existence and the localization of internal transport barriers for Hamiltonian systems with periodical perturbation in one-and-a-half degrees of freedom. We systematically study the influence of the parameters which define the unperturbed Hamiltonian on the transport properties. The results are applied for the study of the magnetic transport in tokamaks (toroidal devices used for obtaining energy through controlled thermonuclear fusion). In this case the formation of internal transport barriers is crucial for the (desired) plasma confinement because it prevents the radial transport of charged particles.

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1. Introduction

Many important models in astronomy, plasma physics, fluid dynamics, mechanics are non-autonomous Hamiltonian systems. These systems are generically non-integrable and exhibit chaotic dynamics.

In this paper we focus on non-integrable 1 1/2 degrees of freedom Hamiltonian systems with a broad perturbation spectrum. We are particularly interested on the existence of invariant surfaces (tori) because they are important tools for understanding the dynamics of these systems. The invariant tori cannot be crossed by orbits starting from outside, so they act as barriers which separate two invariant zones of the space. In the unperturbed system every orbit lives on an invariant torus and the dynamics of the (integrable) system is regular. The complicate dynamics of the perturbed systems is partially due to the destruction of the barriers, which enables the orbit to wander in a larger zone of the phase space.

In order to understand this complex dynamics we shall use the Poincaré map (the first return map). It generates a two dimensional discrete dynamical system which preserves the global properties of the continuous system (symplecticity, invariants of the motion, time-reversal or other specific symmetries) and the local dynamical characteristic of the orbits (regular or chaotic). Because the exact Poincaré map cannot be determined (this is equivalent with solving analytically the system), it will be approximated using the symmetric mapping technique [1].

The existence/destruction of the invariant circles in twist systems (i.e. system generated by a twist Poincaré map) is described by the KAM (Kolmogorov-Arnold-Moser) theory [2], the Aubray-Mather theory [3], Greene's criterion [4], renormalization theory [5, 6]. For the non-twist systems (such as using the magnetic fusion terminology, reversed-shear systems), the adapted KAM theory, Greene's criterion [7] and renormalization theory [8] were used for the study of the transition to chaos. In this paper we study the existence of invariant circles in the general non-twist systems generated by the Poincaré map (the reversed shear condition is not imposed) and we explain the influence of the unperturbed Hamiltonian on their position.

The paper is structured as follows: Basic information about 1 1/2 degrees of freedom Hamiltonian systems with a broad perturbation spectrum and symmetric maps is given in Section 2.; Section 3. contains the main results about the existence of invariant circles; some applications in fusion plasma physics are presented in Section 4., before summarizing the results in Section 5..

2. Hamiltonians with a broad perturbation spectrum and symmetric maps

Symmetric maps are important tools in the study of 1 1/2 degrees of freedom Hamiltonian systems because they are good approximations of the Poincaré map. They run much faster than the small step symplectic numerical integration, but the main advantage is that the mapping models have better accuracy in the study of the chaotic dynamics due to the fact that the accumulation of the round-off errors is reduced. In the specific case of Hamiltonians with a broad perturbation spectrum, $H : [0, 2\pi) \times D \times [0, \infty) \to \mathbb{R}$ with

$$H(\theta, I, t) = H_0(I) + \varepsilon \sum_m H_m(I) \cdot \sum_{n=-M}^{M} \cos(m\theta - nt) \tag{1}$$

with $D \subset \mathbb{R}$ and $2M + 1 >> 1$. One can use the mixed generating function of the new momentum and the old angle $F\left(\theta, \overline{I}\right) = \theta \cdot \overline{I} + H_0\left(\overline{I}\right) + \varepsilon \cdot 2\pi \cdot \sum\limits_m H_m\left(\overline{I}\right) \cdot \cos(m\theta) = \theta \cdot \overline{I} + H_0\left(\overline{I}\right) + \varepsilon \cdot S\left(\theta, \overline{I}\right)$ in order to obtain the symmetric Poincaré map (See [1], p. S20) corresponding to $t_0 = 0$, namely $T_\varepsilon : [0, 2\pi) \times D \to [0, 2\pi) \times D$,

$$T_\varepsilon : \begin{cases} \overline{\theta} = \left(\theta + 2\pi W(X) + \frac{\varepsilon}{2}\frac{\partial S}{\partial X}(\theta, X) + \frac{\varepsilon}{2}\frac{\partial S}{\partial X}\left(\overline{\theta}, X\right)\right)(\mathrm{mod} 2\pi) \\ \overline{I} = I - \frac{\varepsilon}{2}\frac{\partial S}{\partial \theta}(\theta, X) + \frac{\varepsilon}{2}\frac{\partial S}{\partial \theta}\left(\overline{\theta}, X\right) \end{cases} \tag{2}$$

where $X = X(\theta, I)$ is the solution of the implicit equation

$$X = I - \frac{\varepsilon}{2}\frac{\partial S}{\partial \theta}(\theta, X). \tag{3}$$

In the discrete system generated by T_ε, the orbit of $(\theta_0, I_0) \in [0, 2\pi) \times D$ is denoted by $O(\theta_0, I_0) = \{T_\varepsilon^n(\theta_0, I_0) = (\theta_n, I_n), n \in \mathbb{N}\}$. The winding function $W : D \to R$, $W(I) = H_0'(I)$ describes the rotation of the orbits of the unperturbed system (corresponding to $\varepsilon = 0$). In this case the orbit of (θ_0, I_0) is given by $I_n = I_0$, $\theta_n = (\theta_0 + 2\pi n W(I_0))(\mathrm{mod} 2\pi)$, so it is rotated with the constant angle $2\pi W(I_0)$ at each step.

The map T_e is a non-twist map if there is $(\theta_0, I_0) \in [0, 2\pi) \times D$ such that

$$\frac{\partial \overline{\theta}}{\partial I}(\theta_0, I_0) = 0, \tag{4}$$

i.e. the twist property is violated in (θ_0, I_0). There are three curves that are considered important for characterizing the non-twist dynamics:

- the critical twist curve [9] which is formed by the points where the twist condition is not fulfilled: $C_{ct,\varepsilon} : \partial\overline{\theta}/\partial I = 0$.
- the regular curve $C_{reg,\varepsilon} : W'(X) = 0$, which is related to the non-twist properties of the continuous system (which is called non-twist if there is $I_0 \in D$ such that $W'(I_0) = 0$ [12]).
- the shearless curve $C_{0shear,\varepsilon}$, which is closure of the orbit having a rotation number which is a local extremum of all rotation numbers of orbits in the map. The shearless curve exists only for reversed-shear systems, i.e. systems having non monotonous winding function.

For the symmetric Poincaré map (2), (3) the implicit equation of the critical-twist curve is

$$C_{ct,\varepsilon} : 2\pi W'(X) + \frac{\varepsilon}{2}\frac{\partial^2 S}{\partial X^2}(\theta, X) + \frac{\varepsilon}{2}\frac{\partial^2 S}{\partial X^2}\left(\overline{\theta}, X\right) = 0. \tag{5}$$

All the orbits of the points of the critical twist curve form the non-twist annulus $NTA_\varepsilon = \{O(\theta, I),\ (\theta, I) \in C_{ct,\varepsilon}\}$. The non-twist annulus collects all the points with non-twist behavior. In the typical case when the equation $W'(I) = 0$ has a single solution I_0, the regular curve has the explicit equation

$$C_{reg,\varepsilon} : I = I_0 + \frac{\varepsilon}{2} \cdot \frac{\partial S}{\partial \theta}(\theta, I_0). \tag{6}$$

In the unperturbed system, the regular curve, the non-twist curve and the shearless curve (when it exists) coincide. In the perturbed case, these curves are different and a natural question arises on how to relate the dynamical properties of the system to each of them.

In the next section we shall show that the existence of invariant circles with the rotation number closed to I_0 is related to the regular curve.

3. Existence of invariant circles for non-twist symmetric maps

The results obtained for some particular reversed-shear non-twist systems [10] are now extended even in the case of monotonous winding function. The following proposition can be proved using the theorem of Ortega [11].

Proposition 1. *Let us consider the system generated by the map T_ε . If*

i) there is $a \in D$ such that $T_\varepsilon([0, 2\pi) \times \{a\}) \subseteq [0, 2\pi) \times \{a\}$

ii) there is $I_0 \in D$ such that $W'(I_0) = 0$

iii) $2\sum_m H_m'(I) + (I - I_0)\sum_m H_m''(I) \neq 0$ for all $I \in [a, a+2]$

then there is $\varepsilon_0 > 0$ such that , for every $\varepsilon < \varepsilon_0$, the map T_ε has invariant circles intersecting the regular curve $C_{reg,\varepsilon}$.

Proof. In the hypothesis of Proposition 1, there is $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$ the map $T_{\varepsilon,0}$, obtained from (2), (3) for the constant winding function $\overline{W}(I) = W(I_0)$ has invariant circles densely filled by orbits having the rotation number closed to $W(I_0)$. The points $(\theta, I) \in C_{reg,\varepsilon}$ have the same image through T_ε and $T_{\varepsilon,0}$. Indeed, in this case $X = I_0$ (from the definition of the regular curve) and, if we expand $W(X)$ in Taylor series around I_0 we obtain $d(T_\varepsilon(\theta, I), T_{\varepsilon,0}(\theta, I)) = \sqrt{2\pi(W(X) - W(I_0))} = 0$. In the annulus generated by the regular curve, for small enough values of ε, the map T_ε is closed to the map $T_{\varepsilon,0}$ because $d(T_\varepsilon(\theta, I), T_{\varepsilon,0}(\theta, I)) = \sqrt{2\pi(W(X) - W(I_0))} = O(\varepsilon) \approx 0$. In this case the map T_ε has similar properties with $T_{\varepsilon,0}$. In particular it has invariant circles which are filled by orbits having the rotation number closed to $W(I_0)$ for small enough values of ε. □

From Proposition 1 it results that the orbits having the rotation number closed to $W(I_0)$ are contained in the regular annulus. The theorem of Ortega shows that there are infinitely many invariant circles having rotation numbers closed to W(I_0). They cover a zone where the map T_ε is almost integrable (because it is closed to an integrable one) and form a transport barrier with positive area. We call this zone a transport barrier because it cannot be crossed by orbits starting from outside.

It was widely considered that the existence of a transport barrier with positive area is characteristic to non-twist systems with non-monotonous winding functions, but our result shows that it is true even for monotonous winding functions with a critical point, i.e. for a category of non-twist systems. From Proposition 1 it results also that the map T_ε has not invariant circles with the rotation number closed to $W(I_0)$ if all the points of the regular curve $C_{reg,\varepsilon}$ have chaotic orbits.

The existence of a transport barrier with positive area reflects the non-twist property of the discrete system, so it is natural to see also the relation with the critical twist curve. The critical twist curve and the regular curve have common points if there is (θ, X) such that $\frac{\partial^2 S}{\partial X^2}(\theta, X) + \frac{\partial^2 S}{\partial X^2}\left(\overline{\theta}, X\right) = 0$. This happens for small enough values of ε (which is involved in the computation of X and $\overline{\theta}$). In this situation the non-twist annulus and the regular annulus intersect, usually the regular annulus is contained in the non-twist annulus. Proposition 1 shows that the transport barrier is included in the non-twist annulus, but our result is stronger The shearless curve, when it exists, is also contained in the non-twist annulus because it also reflect the non-twist properties of the system. It is considered that the transport barrier contains the shearless curve, numerical simulations confirm this, but Proposition 1 is not related to it, being valid even in cases when the shearless curve does not exist.

In conclusion, we consider that the regular curve is appropriate for studying the existence of invariant circles (transport barriers) in non-twist systems. An other important point is the width of the transport barrier in non-twist systems. It can be characterized using the flatness coefficient [13] of the winding function $f_W(I_0) = \lim_{I \to \widetilde{I}} (\ln|W(I) - W(I_0)|) / (\ln|I - I_0|)$. It can be proved that the transport barrier is larger when the flatness coefficient is larger.

All these results have interesting practical applications.

4. Applications in fusion plasma physics

The symmetric tokamap describes some magnetic configurations that may be encountered in tokamaks. These are toroidal devices used for obtaining the thermo-controlled nuclear fusion. Because the tokamaks are toroidal devices, it is natural to use toroidal coordinates (r, θ, ζ) in order to describe the magnetic field (ζ is the toroidal angle and (r,θ) are the poloidal coordinates in a circular poloidal section. Instead of the poloidal radius r, the toroidal flux $\psi = r^2/2$ is commonly used because it represents a canonical variable [14]. The Hamiltonian of the system, obtained from (1) for $m = 1$, is

$$H_T(\theta,\psi,\zeta) = \int W(\psi)\,d\psi - \frac{\varepsilon}{4\pi^2}\frac{\psi}{\psi+1}\cdot\sum_{n=-M}^{M}\cos(\theta - n\zeta). \qquad (7)$$

In the Hamiltonian (7) the toroidal angle ζ plays the role of time and (ψ,θ) are the action-angle variables. The particular form of the perturbation, $H_1(\psi) = -(4\pi^2)^{-1}\psi/(\psi+1)$ was considered in order to respect the minimal requests of toroidal geometry [14, 15].

The symmetric map obtained from (2),(3) is $T_e : [0,2\pi)\times[0,\infty) \to [0,2\pi)\times[0,\infty)$

$$T_\varepsilon : \begin{cases} \overline{\theta} = \left(\theta + 2\pi\cdot W(X) - \frac{\varepsilon}{4\pi}\frac{1}{(1+X)^2}\left(\cos\theta + \cos\overline{\theta}\right)\right)(\mathrm{mod}2\pi) \\ \overline{\psi} = \psi - \frac{\varepsilon}{4\pi}\frac{X}{1+X}\left(\sin\theta + \sin\overline{\theta}\right) \end{cases} \qquad (8)$$

where X is the positive solution of the equation $X = \psi - \frac{\varepsilon}{4\pi}X/(X+1)\sin\theta$. The map (8) is an area preserving map for which the line $\psi = 0$ is invariant (i.e. $T_e(\theta,0) = (\overline{\theta},0)$). For a general description we will consider the winding function

$$W(\psi) = W_0 - c(\psi - \psi_0)^n \qquad (9)$$

where $n \geq 2$ is a natural number. A detailed analysis of the dynamical properties of the symmetric tokamap involving the winding function (9) can be found in [13].

The system generated by the Hamiltonian (7) satisfies the hypothesis of Proposition 1: the condition i) is fulfilled for $a = 0$, W has a critical point and $2H_1^{'}(\psi) + (\psi - \psi_0)H_1^{''}(\psi) = -(2\pi^2)^{-1}(1+\psi_0)/(1+\psi)^3 \neq 0$ for all $\psi \geqslant 0$. The critical twist curve $C_{ct,\varepsilon} : \pi(1+X)\cdot W'(X) + 2\pi\cdot W(X) + \theta - \overline{\theta} = 0$ and the regular curve

$$C_{reg,\varepsilon} : \psi = \psi_0 + \frac{\varepsilon}{4\pi}\frac{\psi_0}{1+\psi_0}\sin\theta$$

coincide for $\varepsilon = 0$ (i.e. $C_{ct,0} = C_{reg,0} : \psi = \psi_0$) and they have common points, even for large values of ε.

The theoretical results presented in Section 3., applied to the symmetric tokamap, show that:

- invariant circles with the rotation number closed to W_0 exists for small enough values of ε;
- these invariant circles, which form the internal transport barrier denoted $ITB_{W_0,n}$, intersect the regular curve $C_{reg,\varepsilon}$.

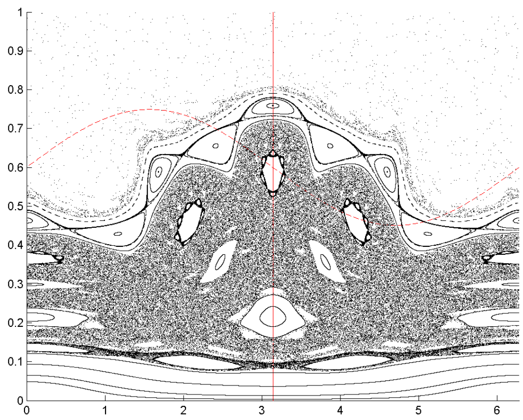

Figure 1: Transport barrier in the symmetric tokamap.

In Figure 1 is presented the phase portrait of the symmetric tokamap corresponding to $\varepsilon = 4$, $W_0 = 5/(2\pi)$, $c = 2$, $\psi_0 = 0.6$, $n = 2$. The dashed line is the regular curve and the transport barrier intersecting it can be observed. Because the winding function has a maximum in ψ_0, two twin island chains with the same rotation number are formed and their reconnection can be observed inside the transport barrier. Proposition 1 can be applied in order to build transport barriers by controlling the winding function. From mathematical point of view the controlled winding function must have a critical point, eventually in a prescribed position.

5. Conclusions

In this paper we presented some general results concerning the existence of invariant tori in 1 1/2 degrees of freedom Hamiltonian systems. The results were obtained using the discrete system generated by the Poincaré map. It was proved that, for a large category of non-twist systems, the invariant tori exist when the amplitude of the perturbation is small enough. It was proved that these invariant tori intersect the regular curve. The transport barrier formed by these invariant tori is robust because in that region the system is almost integrable, being close to an integrable one. A way to build transport barrier by modifying the winding function was also pointed out. The results were exemplified using a Hamiltonian system that describes some magnetic configurations that may be encountered in tokamaks. In this situation the existence of the transport barrier is crucial for plasma's confinement because it prevents the radial motion of the charged particles.

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