Symmetries, Integrability and Exact Solutions for Nonlinear Systems*

R. Cimpoiasu†
University of Craiova, 13 A.I.Cuza, 200585 Craiova, Romania

R. Constantinescu‡
University of Craiova, 13 A.I.Cuza, 200585 Craiova, Romania

Abstract

The paper intends to offer a general overview on what the concept of integrability means for a nonlinear dynamical system and on how the symmetry method can be applied for approaching it. After a general part where key problems as direct and indirect symmetry methods or an optimal system of solutions are tackled with, in the second part of the lecture two concrete models of nonlinear dynamical systems are effectively studied in order to illustrate how the procedure is working. The two models are the $2D$ Ricci flow model coming from general relativity and the $2D$ convective-diffusion equation . Part of the results, especially the ones concerning the optimal systems of solutions, are new ones.

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1. Integrability and symmetries. Key aspects.

1.1. The concept of integrability for dynamical systems

Dynamical systems described by nonlinear partial differential equations are frequently used to model a wide variety of phenomena in physics, chemistry, biology and other fields [1]. The modelling process includes to find solutions of those partial differential equations. If these solutions should exist, the

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† e-mail address: rodicimp@yahoo.com

‡ e-mail address: rconsta@yahoo.com

diffential system is said to be integrable. Sometime it is difficult to find a complete set of solutions and it would be quite enough if one could decide on the integrability of the system. There are many methods which can be used to fulfill this aim: Hirota's bilinear method, the Backlund transformation method, the inverse scattering method, the Lax pair operator, the Painleve analysis and others [2]. Each method has its own significant properties. For example, while the Lax and the Painleve methods are mostly testing the integrability, Hirota's bilinear method is very efficient for the effective determining of the multiple soliton solutions for a wide class of nonlinear evolution equations [3]. As a conclusion, in order to decide that a nonlinear differential equation is integrable, one of the following situations should appear:

(i) the existence of a number of functionally independent first integrals/invariants equal to the order of the system in general and half of that number for a Lagrangian system as a consequence of Liouville's Theorem;

(ii) the existence of a sufficient number of Lie symmetries able to reduce the partial differential equation to an ordinary differential equation;

(iii) the possession of the Painlevé property [4].

In this paper the first two criteria will be investigated.

1.2. The symmetry method for solving dynamical systems

Many natural phenomena are described by a system of nonlinear partial differential equations (pdes) which is often difficult to be solved analytically, since there is no general theory for completely solving nonlinear pdes. One of the most useful techniques for finding exact solutions for the dynamical systems described by nonlinear pdes is *the symmetry method.* On the one hand, we can consider the symmetry reduction of differential equations and thus obtain classes of exact solutions. On the other hand, by definition, a symmetry transforms solutions into solutions, and thus symmetries can be used to generate new solutions from known ones.

Initially, the symmetry method for solving partial differential equations was developed for what is currently known as the *Lie (classical) symmetry method (CSM).* We shall present now a short introduction to this approach [5].

Let us consider a n-th order partial differential system:

$$\Delta_\nu(x, u^{(n)}[x]) = 0, \tag{1}$$

where $x \equiv \{x^i,\ i = \overline{1,p}\} \subset R^p$ represent independent variables, while $u \equiv \{u^\alpha, \alpha = \overline{1,q}\} \subset R^q$ dependent ones. The notation $u^{(n)}$designates the set of variables which includes u and the partial derivatives of u up to the n-th order.

The general infinitesimal symmetry operator has the form:

$$U = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{2}$$

The Lie symmetries represent the set of all the infinitesimal transformations which keep invariant the differential system. If we consider $U^{(n)}$ the extension of the n-th order of (2), the invariance condition can be expressed as:

$$U^{(n)}[\Delta] \mid_{\Delta=0} = 0. \tag{3}$$

The characteristic equations associated to the general symmetry generator (2) have the form:

$$\frac{dx^1}{\xi^1} = ... = \frac{dx^p}{\xi^p} = \frac{du^1}{\phi_1} = ... = \frac{du^q}{\phi_q} \ . \tag{4}$$

By integrating the characteristic system of ordinary differential equations (4), the invariants I_r, $r = \overline{1,(p+q-1)}$ of the analyzed system can be found.

There have been *several generalizations* of the Lie symmetry method which include:

1) the *non-classical symmetry method* (*NSM*) (also referred to as *the conditional method*) of Bluman and Cole [6],
2) the *direct method* of Clarkson and Kruskal [7],
3) the *differential constraint approach* of Olver and Rosenau [8]
4) the *generalized conditional symmetry* method due to Fokas, Liu and Zhdanov [9].

The *direct method* represents a direct, algorithmic, and non group theoretic method for finding symmetry reductions. The relationship between this direct method and the nonclassical method has been discussed in many papers (e.g., [10], [11], [12] and [13]).

The *differential constraint approach* proposed a generalization of the nonclassical method. Its promoters have shown that the original system of partial differential equations can be enlarged by appending additional differential constraints (side conditions), such that the resulting over determined system of partial differential equations should be satisfying the compatibility conditions.

As well, in further efforts to find new symmetries of PDE's which would lead to additional new invariant solutions, much work has been done in the area of higher-order symmetries. In particular, for an evolution equation in two independent variables and one dependent variable, has been introduced in [9] the method of *generalized conditional symmetries (GCS)* or conditional Lie-Bäcklund symmetries.

1.3. An optimal system of solutions

In general, when a differential equation should admit a Lie group $\mathcal{G}_r$ and its Lie algebra $\mathcal{L}_r$ is of dimension $r \rangle 1$, we would to minimize the search for invariant solutions by finding nonequivalent branches of solutions. This should lead to the concept of *optimal system.*

It is well known that, for one-dimensional subalgebras, the problem of finding an optimal system of subalgebras is essentially the same as the problem of classifying the orbits of the adjoint transformations.

In Ovsiannikov [14], the *global matrix of the adjoint transformations* is used for constructing the one-dimensional optimal system.

In Olver [5], a slightly different technique is employed: it consists in constructing a table, named the *adjoint table*, which would present the separate adjoint actions of each element in $\mathcal{L}_r$ as it should act on all the other elements.

The procedure reported in Ruggieri and Valenti [15], is a mixture of the above presented procedures and it consists in constructing the *global matrix of the adjoint transformations* by the means of the *adjoint table.*

One of the advantages of symmetry analysis is the possibility to find solutions of the original pdes by solving odes. These odes, called *reduced equations*, are obtained by introducing suitable new variables, determined as invariant functions, with respect to the infinitesimal generators.

On the basis of the infinitesimal generators of the optimal systems of Lie algebras of the analyzed model, we can construct the reduced odes for the given model and find exact solutions.

1.4. The inverse Lie symmetry problem

Usually, *the direct symmetry problem* of evolutionary equations is considered for the aim of finding their exact solutions. It is also known as the classical symmetry method. Firstly, it consists in determining the Lie symmetry group corresponding to a given evolutionary equation. Then, using the characteristic equations, could be obtained the Lie invariants associated to each symmetry operator. Further, these invariants, following the reduced similarity procedure, should determine the reduced equation which could be solve and generate the similarity solution for the analyzed model.

Also, the *inverse symmetry problem* [16] could be considered. Let us ask the question: what is the largest class of evolutionary equations which would be equivalent from the point of view of their symmetries?. So, this problem could be solved by imposing a concrete symmetry group to a general analyzed model. With this condition, the general symmetry determining equations could be solved and should allow us to determine all the concrete models which admit the same Lie symmetry group.

Let us consider a $2D$ dynamical system described by a second order partial derivative equation of the general form:

$$\begin{aligned} u_t &= A(x,y,t,u)u_{xy} + B(x,y,t,u)u_x u_y + C(x,y,t,u)u_{2x} + D(x,y,t,u)u_{2y} \\ &\quad + E(x,y,t,u)u_y + F(x,y,t,u)u_x + G(x,y,t,u), \end{aligned} \tag{5}$$

with $A(x,y,t,u)$, $B(x,y,t,u)$, $C(x,y,t,u)$, $D(x,y,t,u)$, $E(x,y,t,u)$, $F(x,y,t,u)$, $G(x,y,t,u)$ arbitrary functions of their arguments.

The general expression of the Lie symmetry operator which leaves (5) invariant is:

$$\begin{aligned} U(x,y,t,u) &= \varphi(x,y,t,u)\frac{\partial}{\partial t} + \xi(x,y,t,u)\frac{\partial}{\partial x} \\ &\quad + \eta(x,y,t,u)\frac{\partial}{\partial y} + \phi(x,y,t,u)\frac{\partial}{\partial u}\ . \end{aligned} \tag{6}$$

Through the loss of the generality we could choose in the previous expression $\varphi \equiv 1$.Then, the generator (6) becomes:

$$U(x,y,t,u) = \frac{\partial}{\partial t} + \xi(x,y,t,u)\frac{\partial}{\partial x} + \eta(x,y,t,u)\frac{\partial}{\partial y} + \phi(x,y,t,u)\frac{\partial}{\partial u}\ . \tag{7}$$

Following the symmetry theory [5], the following partial differential system with 11 equations is obtained:

$$\begin{aligned} 0 &= \xi_u,\ 0 = \eta_u, \\ 0 &= B\eta_x - D\phi_{2u},\ 0 = B\xi_y - C\phi_{2u}, \\ 0 &= A\eta_y - \eta A_y - A_u\phi + A\xi_x - \xi A_x + +2D\xi_y + 2C\eta_x - A_t, \\ 0 &= A\eta_x + 2D\eta_y - \eta D_y - \xi D_x - D_u\phi - D_t, \\ 0 &= -A\phi_{2u} + B\xi_x - B\phi_u + B\eta_y - B_t - B_x\xi - B_u\phi - B_y\eta, \\ 0 &= -\eta_t + F\eta_x - B\phi_x + E\eta_y - E_t - E_x\xi - E_y\eta - E_u\phi \\ &\quad + A\eta_{xy} - A\phi_{xu} + C\eta_{2x} + D\eta_{2y} - 2D\phi_{yu}, \\ 0 &= -\xi_t - B\phi_y + F\xi_x + E\xi_y - F_t - F_x\xi - F_y\eta - F_u\phi \\ &\quad -A\xi_{xy} - A\phi_{yu} + C\xi_{2x} + D\xi_{2y} - 2C\phi_{xu}, \\ 0 &= \phi_t + G\phi_u - F\phi_x - E\phi_y - G_t - G_x\xi - G_y\eta - G_u\phi, \\ &\quad -A\phi_{xy} - C\phi_{2x} - D\phi_{2y} \end{aligned} \tag{8}$$

The number of equations and of unknown functions which appear in the system (8) is relatively high. Two approaches are now possible: (i) to find the symmetries of a given evolutionary equation, which means to choose concrete forms for $A(x,y,t,u)$, $B(x,y,t,u)$, $C(x,y,t,u)$, $D(x,y,t,u)$, $E(x,y,t,u)$, $F(x,y,t,u)$, $G(x,y,t,u)$ and to make use of the system (8) in order to find the coefficient functions $\xi(x,y,t)$, $\eta(x,y,t)$ and $\phi(x,y,t,u)$ of the Lie operator; (ii) to solve the system (8) by taking as unknown variables $A(x,y,t,u)$, $B(x,y,t,u)$, $C(x,y,t,u)$, $D(x,y,t,u)$, $E(x,y,t,u)$, $F(x,y,t,u)$, $G(x,y,t,u)$ and by imposing a concrete form for the symmetry group. The first approach represents *the direct symmetry problem* and it is the usual one chosen in the study of the Lie symmetries of a given dynamical system. The second approach, (ii), represents the *inverse symmetry problem* and it is more special, allowing us to determine all the equations which are equivalent from the point of view of the symmetry group they do admit.

2. Applications

In the considerations below we will solve the direct and inverse Lie symmetry problems for two $2D$ nonlinear models: the Ricci flow model and the convective-diffusion equation.

2.1. The Lie symmetry problems for the 2D Ricci flow model

One of the most fruitful models used in the study of the black holes and in the attempt of obtaining a quantum theory of gravity is connected with the *Ricci flow equations* [17].

We will investigate a $2D$ model for the Ricci flow equation, a nonlinear parabolic equation obtained when the components of the metric tensor $g_{\alpha\beta}$ should be deformed according to the equation:

$$\frac{\partial}{\partial t}g_{\alpha\beta} = -R_{\alpha\beta}, \tag{9}$$

where $R_{\alpha\beta}$ is the Ricci tensor for the n-dimensional Riemann space. The metric tensor of the space $g_{\alpha\beta}$ will be connected with the Riemann metric in the conformal gauge:

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = \frac{1}{2}\exp\{\Phi(X,Y,t)\}(dX^2 + dY^2). \tag{10}$$

The "potential" $\Phi(X,Y,t)$ satisfies the equation:

$$\frac{\partial}{\partial t}e^\Phi = \triangle\Phi. \tag{11}$$

It has been noticed in [18] that the equation (11) is pretty similar with the Toda equation describing the integrable interaction of a collection of two-dimensional fields $\{\Phi_i, i = 1,2\}$ coupled by a Cartan matrix (K_{ij}):

$$\sum_j K_{ij}e^{\Phi_j(X,Y)} = \triangle\Phi_i(X,Y). \tag{12}$$

Introducing the field $u(x,y,t)$ given by

$$u(x,y,t) = e^\Phi, \tag{13}$$

the equation (11) takes the form:

$$u_t = (\ln u)_{xy}. \tag{14}$$

An equivalent form for the previous equation, which will be used in the following considerations of the paper, is:

$$u_t = \frac{u_{xy}}{u} - \frac{u_x u_y}{u^2} \ . \tag{15}$$

The previous equation could be derived from the general one (5) by the choice of the following particular coefficient functions:

$$\begin{aligned} &A(x,y,t,u) = \frac{1}{u}, B(x,y,t,u) = -\frac{1}{u^2} \ , \\ &C(x,y,t,u) = D(x,y,t,u) = E(x,y,t,u) = F(x,y,t,u) = G(x,y,t,u) \equiv 0. \end{aligned} \tag{16}$$

As we proved in [19], the equation (15) admits the 4−dimensional Lie algebra spanned by the independent operators shown below:

$$V_1 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} \ , \quad V_2 = \frac{\partial}{\partial x} \ , \quad V_3 = y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} \ , \quad V_4 = \frac{\partial}{\partial y} \ . \tag{17}$$

The forms of the operators V_i, $i = \overline{1,4}$ suggest their significances: V_2, V_4 do generate the symmetry of space translations, while V_1, V_3 are associated with the scaling transformations.

When the Lie algebra of these operators is computed, the only non-vanishing relations are:

$$[V_2, V_1] = V_2, [V_4, V_3] = V_4. \tag{18}$$

2.1.1. An optimal system of subalgebras for the 2D Ricci flow model

It is well known that reducing the independent variables by one would be possible by using any linear combination of the generators of symmetry (17) $V_i, i = \overline{1,4}$. We will construct a set of minimal combinations known as an optimal system [5]. In order to construct the optimal system we need the commutators of the admitted symmetries given in the Table 1.

$[V_i, V_j]$	V_1	V_2	V_3	V_4
V_1	0	$-V_2$	0	0
V_2	V_2	0	0	0
V_3	0	0	0	$-V_4$
V_4	0	0	V_4	0

Table 1. Lie brackets of the admitted symmetry algebra

An optimal system of a Lie algebra is a set of $l-$dimensional subalgebras such that each $l-$dimensional subalgebra should be equivalent to a an unique element of the set under some element of the adjoint representation. The adjoint representation is constructed using the formula [5]:

$$Ad(\exp(\varepsilon V_i))V_j = \sum \frac{\varepsilon^n}{n!}(adV_i)^n V_j = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2!}[V_i, [V_i, V_j]] - ... \tag{19}$$

Let us consider the linear combination of the symmetry generators:

$$V = a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4. \tag{20}$$

Our task is to simplify as many of the coefficients a_i as possible through judicious applications of adjoint maps to V. Suppose first that $a_1 \neq 0$ in (20). We may re-scale a_1 so that $a_1 = 1$. Let us start by the combination:

$$V^{(1)} = V_1 + a_2V_2 + a_3V_3 + a_4V_4. \tag{21}$$

If we should act on $V^{(1)}$ by $Ad(\exp(a_2V_2))$, we could make the coefficient of V_2 vanish:

$$V^{(2)} = V_1 + a_3V_3 + a_4V_4. \tag{22}$$

Next, we should act on $V^{(2)}$ by $Ad(\exp(\frac{a_4}{a_3}V_4))$ to cancel the coefficient of V_4, leading to the operator:

$$V^{(3)} = V_1 + a_3V_3. \tag{23}$$

By using the adjoint representation (19), no further simplification would be possible. Consequently, the $1-$dimensional subalgebra spanned by V with $a_1 \neq 0$ is equivalent to the one spanned by $V_1 + \beta V_3$, $\beta \in R$.

The remaining $1-$dimensional subalgebras are spanned by operators with $a_1 = 0$ which have the expressions:

$$V^{(4)} = a_2V_2 + a_3V_3 + a_4V_4. \tag{24}$$

Let us assume that $a_2 \neq 0$ and let us scale to make $a_2 = 1$. Now we should act on $V^{(4)}$ by $Ad(\exp(\frac{a_4}{a_3}V_4))$ so that it would be equivalent with the operator:

$$V^{(5)} = V_2 + a_3V_3. \tag{25}$$

No further simplification is possible. Consequently, the $1-$dimensional subalgebra spanned by V with $a_2 \neq 0$ is equivalent to the one spanned by $V_2 + \alpha V_3$, $\alpha \in R$.

If we should consider the case $a_1 = a_2 = a_3 = 0$, $a_3 \neq 0, a_3 = 1$, the following generator would be obtained:

$$V^{(6)} = V_3 + a_4 V_4. \tag{26}$$

Through acting on $V^{(6)}$ by $Ad(\exp(a_4 V_4))$, we would obtain the operator V_3 which represents the next subalgebra of the optimal system.

Finally, let us consider the last case $a_1 = a_2 = a_3 = 0$, $a_4 \neq 0, a_4 = 1$ in (20). The last subalgebra V_4 results from it.

As a conclusion, the optimal system of $1-$dimensional subalgebras has the form:

$$\{V_2 + \alpha V_3, V_1 + \beta V_3, V_3, V_4\}. \tag{27}$$

2.1.2. Invariant solutions for the $2D$ Ricci flow

Let us pass now to the problem of the *invariant quantities.* We shall analyze the invariants associated with the optimal system of symmetry operators (27).

- The operator $V_2 + \alpha V_3$ from (27) has the characteristic equations:

$$\frac{dt}{0} = \frac{dx}{1} = \frac{dy}{\alpha y} = \frac{du}{-\alpha u}\,. \tag{28}$$

By integrating these equations 3 invariants should result, with the expressions:

$$I_1 = t, \qquad I_2 = y e^{-\alpha x}, \qquad I_3 = yu\,. \tag{29}$$

By introducing the similarity variable $z \equiv I_2 = y e^{-\alpha x}$, designating the invariant $I_3 = h(t,z)$ as a function of the other ones, the following solution is obtained:

$$u(t,x,y) = \frac{h(t,z)}{y}\,. \tag{30}$$

Setting the derivatives of (30) into the Ricci equation (15), the similarity reduced equation for $h(t,z)$ results with the form:

$$h_t h^2 - \alpha z^2 h h_{2z} - \alpha z^2 h_z^2 + \alpha z h h_z = 0\,. \tag{31}$$

The solution of the previous equation is:

$$h(t,z) = -\frac{1}{2}\left(r_3 t + \frac{r_2 r_3}{2 r_1}\right)\left(-1 + \tanh^2\left(\frac{\sqrt{\alpha r_3}(r_4 - \ln z)}{2\alpha}\right)\right), \tag{32}$$

with α, r_1, r_2, r_3 arbitrary constants and z the similarity variable.

Consequently, the invariant solution corresponding to the operator $V_2+\alpha V_3$ has the final form:

$$u(t,x,y) = -\frac{1}{2y}\left(r_3 t + \frac{r_2 r_3}{2r_1}\right)\left(-1+\tanh^2\left(\frac{\sqrt{\alpha r_3}(r_4-\alpha x+\ln y)}{2\alpha}\right)\right). \tag{33}$$

- The operator $V_1+\beta V_3$ from (27) has the characteristic equations:

$$\frac{dt}{0} = \frac{dx}{x} = \frac{dy}{\beta y} = \frac{du}{-(1+\beta)u}\,. \tag{34}$$

  In this second case, following the same procedure, we obtain, similarly, 3 independent invariants with the expressions:

$$I_1 = t, \qquad I_2 = yx^{-\beta}, \qquad I_3 = y^{(1+\beta)/\beta}u\,. \tag{35}$$

  By introducing the similarity variable $z \equiv I_2 = yx^{-\beta}$, designating the invariant $I_3 = h(t,z)$ as a function of the other ones, the following solution is obtained:

$$u(t,x,y) = h(t,z)y^{-(1+\beta)/\beta}. \tag{36}$$

Setting the derivatives of (36) into the Ricci equation (15), the following $(1+1)$ reduced equation for $h(t,z)$ results:

$$h_t h^2 z^{(-1/\beta-2)} + \beta h h_{2z} + \beta z^{-1} h h_z - \beta h_z^2 = 0\,. \tag{37}$$

The solution of the previous equation is:

$$h(t,z) = -\frac{(p_1 t + p_2)}{2p_3^2 p_1 \beta} z^{1/\beta}\left(-1+\tanh^2\left(\frac{p_4\beta - \ln(z)}{2p_3\beta}\right)\right), \tag{38}$$

with $\beta, p_1, p_2, p_3, p_4$ arbitrary constants and z the similarity variable.

Consequently, by using (36) the invariant solution corresponding to the operator $V_1+\beta V_3$ has the final form:

$$u(t,x,y) = -\frac{1}{2p_3^2 p_1\beta}\frac{(p_1 t+p_2)}{xy}\left(-1+\tanh^2\left(\frac{p_4\beta - \ln y + \beta\ln(x)}{2p_3\beta}\right)\right). \tag{39}$$

- Because (15) is symmetric in respect to x and y, a second similarity solution of the form:

$$u(x,y) = \frac{g_3(x)}{y}\,, \qquad \forall g_3(x), \tag{40}$$

  exists, which is generated by the symmetry operator V_3 from (17).
- Again, for the reason of symmetry in x and y of the analyzed model (15), the last similarity solution, associated to the symmetry operator V_4 from (17), is generated as below:

$$u(x) = g_4(x)\,, \qquad \forall g_4(x). \tag{41}$$

2.2. Lie symmetry problems for the 2D convective-diffusion equation

The second nonlinear application is represented by the $2D$ convective-diffusion equation [20]. It is a parabolic partial differential equation, which describes physical phenomena where particles or energy (or other physical quantities) are transferred inside a physical system through two processes: diffusion and convection. In the simpler case when the diffusion coefficient is variable, the convection velocity is constant and there are no sources or sinks, the equation takes the form:

$$u_t = uu_{2x} + uu_{2y} - vu_x \,, \tag{42}$$

where the diffusion coefficient u and convective velocity $v = \mathit{const}.$ belong to the Ox direction.

It is easy to remark that (42) results from the general class of equations (5) by choosing the particular functions:

$$\begin{aligned} &C(x,y,t,u) = D(x,y,t,u) = u,\ F(x,y,t,u) = -v, \\ &A(x,y,t,u) = B(x,y,t,u) = E(x,y,t,u) = G(x,y,t,u) \equiv 0\,. \end{aligned} \tag{43}$$

2.2.1. Lie symmetries for the 2D convective-diffusion equation

Under the conditions (43) the general determining system (8) for symmetries becomes:

$$\begin{aligned} \phi_{2u} &= 0, \\ \xi_y + \eta_x &= 0, \\ 2u\xi_x - \phi &= 0, \\ 2u\eta_y - \phi &= 0, \\ -\eta_t - v\eta_x + u\eta_{2x} + u\eta_{2y} - 2u\phi_{yu} &= 0, \\ -\xi_t - v\xi_x + u\xi_{2x} + u\xi_{2y} - 2u\phi_{xu} &= 0, \\ \phi_t + v\phi_x - u\phi_{2x} - u\phi_{2y} &= 0. \end{aligned} \tag{44}$$

It has the solution:

$$\xi = \frac{c_1}{2}(x - vt) + c_2 y + c_3, \qquad \eta = \frac{c_1}{2} y - c_2(x - vt) + c_4, \qquad \phi = c_1 u\,. \tag{45}$$

In this case, the Lie symmetry generator takes the form:

$$\begin{aligned} U(x,y,t,u) = {}& \frac{\partial}{\partial t} + \left(\frac{c_1}{2}(x - vt) + c_2 y + c_3\right)\frac{\partial}{\partial x} \\ &+ \left(\frac{c_1}{2} y - c_2(x - vt) + c_4\right)\frac{\partial}{\partial y} + c_1 u \frac{\partial}{\partial u}. \end{aligned} \tag{46}$$

Consequently, the nonlinear convective-diffusion equation (42) admits the 4−dimensional Lie algebra spanned by the operators shown below:

$$V_1 = \left(\frac{x - vt}{2}\right)\frac{\partial}{\partial x} + \left(\frac{y}{2}\right)\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}\ , \tag{47}$$
$$V_2 = y\frac{\partial}{\partial x} - (x - vt)\frac{\partial}{\partial y}\ , \quad V_3 = \frac{\partial}{\partial x}\ , \quad V_4 = \frac{\partial}{\partial y}\ .$$

When the Lie algebra of these operators is computed, the only non-vanishing relations are:

$$[V_3, V_1] = \frac{1}{2}V_3, \quad [V_4, V_1] = V_4, \quad [V_2, V_3] = V_4, \quad [V_4, V_2] = V_3. \tag{48}$$

2.2.2. An optimal system for the convective-diffusion equation

For this model the commutators of symmetry operators (47) are given below in Table 2:

$[V_i, V_j]$	V_1	V_2	V_3	V_4
V_1	0	0	$-V_3/2$	V_4
V_2	0	0	V_4	$-V_3$
V_3	$V_3/2$	$-V_4$	0	0
V_4	V_4	V_3	0	0

Table 2. Lie brackets of the admitted symmetry algebra

Let us consider the linear combination of the symmetry generators:

$$V = b_1V_1 + b_2V_2 + b_3V_3 + b_4V_4. \tag{49}$$

Our task is to simplify as many of the coefficients b_i as possible through judicious applications of adjoint maps to V.Let us firstly suppose that $b_1 \neq 0$ in (49). We also may re-scale b_1 such that $b_1 = 1$. Let us start with the combination:

$$V^{(1)} = V_1 + b_2V_2 + b_3V_3 + b_4V_4. \tag{50}$$

If we should act on $V^{(1)}$ by $Ad(\exp(2b_3V_3))$, we could make the coefficient of V_3 vanish and we could obtain the operator:

$$V^{(2)} = V_1 + b_2V_2 + b_4'V_4,\ b_4' = b_4 + 2b_2b_3. \tag{51}$$

By using the adjoint representation (19) for our model, no further simplification is possible. Consequently, the 1−dimensional subalgebra spanned by V with $b_1 \neq 0$ is equivalent to the one spanned by $V_1 + \alpha V_2 + \beta V_4$, $\forall$ $\alpha, \beta \in R$.

The remaining 1−dimensional subalgebras are spanned by operators with $b_1 = 0$, which have the forms:

$$V^{(3)} = b_2V_2 + b_3V_3 + b_4V_4. \tag{52}$$

Let us assume that $b_2 \neq 0$ and scale to make $b_2 = 1$. Now we act on $V^{(3)}$ by $Ad(\exp(b_3V_4))$ so that it could become equivalent with the operator:

$$V^{(4)} = V_2 + b_4V_4. \tag{53}$$

Here, further simplification is yet possible. We should act on $V^{(4)}$ by $Ad(\exp(-b_4V_3))$. In this case, we would obtain the operator V_2 which is the following 1−dimensional subalgebra spanned by V with $b_1 = 0$ and $b_2 = 1$.

If we should consider the case $b_1 = b_2 = 0$, $b_3 \neq 0, b_3 = 1$, the following generator would be obtained:

$$V^{(3)} = V_3 + b_4V_4. \tag{54}$$

If we should act on $V^{(3)}$ by $Ad(\exp(\varepsilon V_2))$, where ε is the solution of the equation

$$\frac{b_4}{2}\varepsilon^2 + \varepsilon - b_4 = 0, \tag{55}$$

we vanish the coefficient of V_4 and we could obtain the operator:

$$V^{(4)} = b_3'V_3, \qquad b_3' = 1 + b_4\varepsilon - \frac{\varepsilon^2}{2}\ , \tag{56}$$

where ε verified (55).

Consequently, by the choice of $b_1 = b_2 = 0, b_3 = 1$ in (49), V_3 is generated namely the last subalgebra of the optimal system.

As a conclusion, the optimal system of 1−dimensional subalgebras for the $2D$ convective-diffusion equation is:

$$\{V_2, V_3, V_1 + \alpha V_2 + \beta V_4, \forall\alpha, \beta \in R\}. \tag{57}$$

2.2.3. Invariant solutions for the convective-diffusion equation

Through the reduced similarity method, each operator $\{V_i,\ i = \overline{1,4}\}$ could generate invariant solutions of the model. Let us illustrate for our case what are the concrete forms of the similarity solutions generated not by this base of operators, but by the set of the optimal $1D$ subalgebras (57).

- Taking into account (47), the symmetry operator V_2 has the characteristic equations:

$$\frac{dt}{0} = \frac{dx}{y} = \frac{dy}{vt - x} = \frac{du}{0}\ . \tag{58}$$

In this second case, following the same procedure, we obtain also 3 independent invariants with the forms:

$$I_1 = t, \quad I_2 = vtx - \frac{x^2}{2} - \frac{y^2}{2} , \quad I_3 = u. \tag{59}$$

With the notation $I_2 \equiv z$ and $I_3 = u \equiv g(t,z)$, the reduced equation for $g(t,z)$ will appear:

$$g_t + (2z - v^2t^2)g\, g_{2z} + 2g\, g_z + v^2 t g_z = 0. \tag{60}$$

It admits the solution:

$$g(t,z) = \frac{2z - v^2t^2 + 2q_1}{4t + 2q_2} , \tag{61}$$

where q_1, q_2, v are arbitrary constants.

Thereby, the second similarity solution corresponding to the operator V_2 has the final form:

$$u(t,x,y) = \frac{2vtx - x^2 - y^2 - v^2t^2 + 2q_1}{4t + 2q_2} . \tag{62}$$

- The operator V_3 from (47) yields the characteristic equations:

$$\frac{dt}{0} = \frac{dx}{1} = \frac{dy}{0} = \frac{du}{0} . \tag{63}$$

Here too 3 invariants are generated:

$$I_1 = t, \quad I_2 = y, \quad I_3 = u. \tag{64}$$

Once again, by expressing the last invariant I_3 as a function of the other ones, we obtain the third similarity solution:

$$u(t,y) = \frac{\frac{q_1}{2}y^2 + q_3 y + q_4}{q_2 - q_1 t}, \tag{65}$$

with q_1, q_2 arbitrary constants.

- Again on the ground of (47), the last operator from (57), $V_1 + \alpha V_2 + \beta V_4$, has the characteristic equations:

$$\frac{dt}{0} = \frac{dx}{\alpha y + \frac{x - vt}{2}} = \frac{dy}{\frac{y}{2} + \alpha(vt - x) + \beta} = \frac{du}{u} . \tag{66}$$

By integrating these equations we obtain 3 invariants with the expressions:

$$I_1 = t,\ I_2 = \frac{\frac{y}{2} + \alpha(vt - x) + \beta}{\frac{x}{2} + \alpha y - \frac{vt}{2}}, \qquad I_3 = \frac{u}{\left[\frac{y^2}{2} + \alpha(vt - x) + \beta\right]^2}. \quad (67)$$

By introducing the similarity variable $z \equiv I_2$, designating the invariant $I_3 = h(t, z)$ as a function of the other ones, the following solution is obtained:

$$u(t, x, y) = h(t, z)\left[\frac{y^2}{2} + \alpha(vt - x) + \beta\right]^2. \quad (68)$$

By setting the derivatives of (68) into the convective-diffusion equation (42), we obtain the following $(1 + 1)$ reduced equation for $h(t, z)$:

$$h_t - 2\left(\alpha^2 + \frac{1}{4}\right) z^3 h h_z - 4\left(\alpha^2 + \frac{1}{4}\right) z h h_z - 2\left(\alpha^2 + \frac{1}{4}\right) h^2 = 0. \quad (69)$$

The solution of the previous equation is:

$$h(t, z) = \frac{-1}{2\left(\alpha^2 + \frac{1}{4}\right) t - \gamma}, \quad (70)$$

with α, γ arbitrary constants.

Making use of (68), the invariant solution generated by the operator $V_1 + \alpha V_2 + \beta V_4$ is pointed out:

$$u(t, x, y) = -\frac{1}{2\left(\alpha^2 + \frac{1}{4}\right) t - \gamma}\left[\frac{y^2}{2} + \alpha(vt - x) + \beta\right]^2, \quad (71)$$

where α, β, γ, v arbitrary constants.

2.2.4. The inverse symmetry problem for the $2D$ convective-diffusion equation

Our aim is now to find the class of equations with the generical form (5) which admits the same symmetries as those corresponding to the $2D$ nonlinear convective-diffusion equation (42). Consequently, we have to impose that the coefficient functions (45) which determine the basis of the symmetry operators (47) should verify the general determining system (8).

The solutions of the differential system (8) describe the coefficient functions of the general evolutionary equation (5) as follows:

$$\begin{aligned} A &= B = 0,\ C = D = c_3 u, \\ E(u) &= \sqrt{u}\left[c_4 \cos\left(\frac{c_2}{c_1}\ln(u)\right) - c_5 \sin\left(\frac{c_2}{c_1}\ln(u)\right)\right], \\ F(u) &= \sqrt{u}\left[c_4 \sin\left(\frac{c_2}{c_1}\ln(u)\right) + c_5 \cos\left(\frac{c_2}{c_1}\ln(u)\right) - v\right], \\ G(u) &= c_6 u. \end{aligned} \tag{72}$$

where c_j, $j = \overline{1,6}$ and v are arbitrary constants.

In particular, for $c_3 = 1$, $c_4 = c_5 = c_6 = 0$ and for arbitrary c_1 and c_2, the solution (72) generates the $2D$ nonlinear convective-diffusion equation (42) discussed above.

3. Conclusions

This paper intended to present some key aspects about how a dynamical system described by a nonlinear differential equation in its evolution could be studied by using the symmetry method. The main steps which have to be done in order to find a set of exact solutions are: (*i*) determining of the general form for the symmetry operator; (*ii*) determining of the optimal set of independent operators which could generate the minimal subalgebras; (*iii*) using the optimal set of independent operators and applying the similarity reduction procedure, a complete set of invariant solutions could be generated; (*iv*) last but not least, a special method could be applied in order to find the largest class of nonlinear differential equations which should belong to the same class as a given equation in the sense of the symmetries they observe. This algorithm was applied for two important examples of nonlinear $2D$ partial derivative equations, the Ricci flow and the convective-diffusion equation. For the first example, the optimal system of subalgebras contains the same number of generators, four, as the whole symmetry algebra. The optimal system of symmetry subalgebras for the convective-diffusion equation has the dimension three, in spite of the existence of four independent symmetry operators. In both cases, the whole set of invariant solutions has been obtained.

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