

# From Supergravity to Pure Spinors

Martin Cederwall\*

Fundamental Physics, Chalmers University of Technology  
SE 412 96 Göteborg, SWEDEN

## ABSTRACT

In this talk, we review how the superspace formulation of maximally supersymmetric field theories (including supergravity) naturally leads to introduction of pure spinors and pure spinor superfields, and why the formalism provides off-shell formulations. This approach to pure spinor superfields thus stresses field-theoretic aspects rather than the first-quantised ones normally used *e.g.* in superstring theory. We discuss how the BRST operator arises and the principles behind constructions of actions, as well as the general Batalin–Vilkovisky framework.  $D = 11$  supergravity and its recently constructed supersymmetric action [1] is taken as an example throughout the talk. This is the written version of a lecture given at the 6th Mathematical Physics Meeting, Belgrade, September 2010.

Maximally supersymmetric models<sup>1</sup> have fields that come in on-shell supermultiplets. The supersymmetry algebra on the component fields close (together with gauge transformations) only modulo equations of motion. In a traditional superfield formulation, this is a problem, since it implies that supersymmetry can not be manifested in an action formulation.

For some time, it has been known that the introduction of pure spinors can solve this problem. In fact, it is turned into an advantage. Such a formulation does not contradict any no-go theorems against the existence of auxiliary fields, since the number of component fields added by the introduction of more bosonic variables is infinite. In this talk, I will review the quite natural transition from a traditional superspace formulation of a maximally supersymmetric model to a formalism with pure spinors, and also discuss some formal developments. The discussion will, apart from some final remarks, concern classical field theory, even if one of the eventual goals will be to examine quantum properties of the models in question, with as much symmetry as possible manifest. Some aspects will be touched on only briefly, and in case I am not able to convey the method in a convincing way, more information can be found in the references.

---

\* e-mail address: martin.cederwall@chalmers.se

<sup>1</sup>This means 8 real supersymmetries for scalar multiplets, 16 for vector/tensor multiplets and 32 for supergravity multiplets

There is a close relation between supermultiplets and pure spinors. The algebra of covariant fermionic derivatives in flat superspace is generically of the form

$$\{D_\alpha, D_\beta\} = -T_{\alpha\beta}{}^c D_c = -2\gamma_{\alpha\beta}^c D_c. \quad (1)$$

If a bosonic spinor  $\lambda^\alpha$  is *pure*, *i.e.*, if the vector part  $(\lambda\gamma^a\lambda)$  of the spinor bilinear vanishes, the operator

$$Q = \lambda^\alpha D_\alpha \quad (2)$$

becomes nilpotent, and may be used as a BRST operator. This is, schematically, the starting point for pure spinor superfields. (The details of course depend on the actual space-time and the amount of supersymmetry. The pure spinor constraint may need to be further specified. Eq. (1) may also contain more terms, due to super-torsion and curvature.) The cohomology of  $Q$  will consist of supermultiplets, which in case of maximal supersymmetry are on-shell. The idea of manifesting maximal supersymmetry off-shell by using pure spinor superfields  $\Psi(x, \theta, \lambda)$  is to find an action whose equations of motion is  $Q\Psi = 0$ .

The fact that pure spinors had a rôle to play in maximally supersymmetric models was recognised early by Nilsson [2] and Howe [3, 15]. Pure spinor superfields were developed with the purpose of covariant quantisation of superstrings by Berkovits [4, 7, 20, 10] and the cohomological structure was independently discovered in supersymmetric field theory and supergravity, originally in the context of higher-derivative deformations [5, 6, 8, 9, 11, 24, 13, 14]. The present lecture only deals with pure spinors for maximally supersymmetric field theory.

The canonical example taken to illustrate the mechanisms at play is  $D = 10$  super-Yang–Mills theory. In this lecture, I will take the opportunity to use a supergravity theory,  $D = 11$  supergravity [21] as the example. In a sense, this is the only model that fits our requirements. If we look for a maximal supergravity, the choice is between this model, and type IIB supergravity in  $D = 10$ , or their dimensional reductions. Type IIB contains a self-dual tensor field, which prevents a Lagrangian formulation. So, the choice is  $D = 11$  supergravity; there is no “toy model”. The situation is a bit more technically complicated than for  $D = 10$  SYM, but I hope you will bear with this. The structure turns out to be very rewarding.

The component fields of  $D = 11$  supergravity are

metric	$g_{mn}$	(bosonic)
3-form	$C_{mnp}$	(bosonic)
gravitino	$\psi_m^\alpha$	(fermionic)

The component action takes the form

$$S = \frac{1}{2\kappa^2} \int d^{11}x \sqrt{-g} \left( R - \frac{1}{48} H^{mnpq} H_{mnpq} \right) + \frac{1}{12\kappa^2} \int C \wedge H \wedge H + \text{terms with fermions}, \quad (3)$$

where  $H = dC$  is the 4-form field strength.

The superspace formulation of  $D = 11$  supergravity is well known [22]. It follows the standard procedure for supergravity in superspace. The coordinates  $x^m$  are complemented by fermionic coordinates  $\theta^\mu$ , and we write  $Z^M = (x^m, \theta^\mu)$ . The vielbein (frame) 1-form is extended to a 1-form on superspace with a flat tangent index:

$$E^A = dZ^M E_M^A, \tag{4}$$

$A = (a, \alpha)$  being the flat index. The spin connection 1-form  $\Omega_A{}^B$  is Lorentz valued. One also defines torsion and curvature 2-forms

$$T^A = DE^A = dE^A + E^B \wedge \Omega_B{}^A, \quad R_A{}^B = d\Omega_A{}^B + \Omega_A{}^C \wedge \Omega_C{}^B, \tag{5}$$

which leads to the Bianchi identities

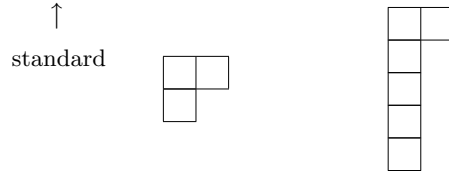
$$DT^A = E^B \wedge R_B{}^A, \quad DR_A{}^B = 0. \tag{6}$$

In Einstein (bosonic) gravity, torsion is set to zero. This does not happen here, as we will see shortly.

Remember that all components of the vielbein and spin connection are superfields. We have much too many fields. Generically one only needs the lowest-dimensional superfield, in this case  $E_\mu{}^\alpha$ , which has (inverse length) dimension  $-\frac{1}{2}$ . All other superfields will be related to it, and it will contain all the physical component fields. The method for eliminating other superfields as independent degrees of freedom is by using *conventional constraints*. They are of two types: those eliminating the spin connection and those eliminating (part of) the vielbein. I will not describe the transformations used in order to implement the conventional constraints; a detailed account can be found in refs. [23, 24]. The transformations are such that the transformed fields satisfy the Bianchi identities if the original ones do.

Conventional constraint should be implemented at the level of “field strengths” – in this case on the torsion. Systematically applying the associated transformations, it turns out that the torsion can always be brought to the form

$$T_{\alpha\beta}{}^c = 2\gamma_{\alpha\beta}^c + \frac{1}{2} U^c{}_{e_1 e_2} \gamma_{\alpha\beta}^{e_1 e_2} + \frac{1}{5!} V^c{}_{e_1 \dots e_5} \gamma_{\alpha\beta}^{e_1 \dots e_5} \tag{7}$$



The tensor superfields  $U$  and  $V$  are all that is left in the torsion at dimension 0. The Young tableaux indicate the irreducible  $so(1, 10)$  modules with this symmetry, which in Dynkin notation will be labelled (11000) and

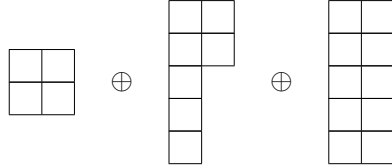
(10002), respectively. Sofar, the fields remain off-shell. It is known that demanding  $U = V = 0$ , *i.e.*, taking the torsion at dimension 0 to have the “standard” form of eq. (1), implies the equations of motion. Demanding  $U = V = 0$  is a *physical constraint*, as opposed to a conventional one. There is no guarantee that such a constraint does not interfere with the Bianchi identities, these being integrability conditions on the torsion. Indeed one finds, by systematically solving the torsion Bianchi identity, that the equations of motion are forced on the component fields.

All physical fields are, as mentioned, contained in the supergeometry. For example, the 4-form field strength is found at dimension 1 as

$$T_{a\beta}{}^\gamma \propto H_{ae_1e_2e_3}(\gamma^{e_1e_2e_3})_\beta{}^\gamma - \frac{1}{8}H^{e_1e_2e_3e_4}(\gamma_{ae_1e_2e_3e_4})_\beta{}^\gamma. \quad (8)$$

See *e.g.* ref. [24] for details. Of course, the vielbein can contain the 3-form  $C$ , which is not gauge invariant, only through its field strength.

There is also a closed superspace 4-form, which contains the bosonic, physical, one. The construction of the super-4-form relies on supergeometric data (the torsion), so this is not an independent construction. However,  $C_{\alpha\beta\gamma}$  contains the entire linearised supermultiplet, and the linearised equations of motion are obtained by demanding that the irreducible modules



in  $H_{\alpha\beta\gamma\delta}$  vanish (the rest are conventional constraints).

We note that the interesting modules both in  $T$  and  $H$  are ones containing columns with 2 and 5 boxes. This is of course no coincidence. They come from pairs of fermionic indices on the components in fermionic directions of superspace forms:  $T_{\alpha\beta}{}^a$  and  $H_{\alpha\beta\gamma\delta}$ . Fermionic form indices are symmetrised, and the symmetric product of two spinors in  $D = 11$  contains a 1-form (vector), a 2-form and a 5-form. Roughly speaking, the vector part goes away by the conventional constraints, and the rest remains.

$$\begin{array}{ccc}
 E_\alpha{}^a : & \boxed{a}{}_\alpha & \text{or} & C_{\alpha\beta\gamma} : & \boxed{\phantom{a}}{}_\alpha \oplus \begin{array}{c} \boxed{\phantom{a}} \\ \boxed{\phantom{a}} \\ \boxed{\phantom{a}} \end{array} \\
 \downarrow & & & \downarrow & \\
 T_{\alpha\beta}{}^a : & \begin{array}{c} \boxed{\phantom{a}} \quad \boxed{a} \\ \boxed{\phantom{a}} \end{array} \oplus \begin{array}{c} \boxed{\phantom{a}} \\ \boxed{\phantom{a}} \\ \boxed{\phantom{a}} \\ \boxed{\phantom{a}} \end{array} & & H_{\alpha\beta\gamma\delta} : & \begin{array}{c} \boxed{\phantom{a}} \quad \boxed{\phantom{a}} \\ \boxed{\phantom{a}} \end{array} \oplus \begin{array}{c} \boxed{\phantom{a}} \quad \boxed{\phantom{a}} \\ \boxed{\phantom{a}} \end{array} \oplus \begin{array}{c} \boxed{\phantom{a}} \quad \boxed{\phantom{a}} \\ \boxed{\phantom{a}} \quad \boxed{\phantom{a}} \\ \boxed{\phantom{a}} \quad \boxed{\phantom{a}} \\ \boxed{\phantom{a}} \quad \boxed{\phantom{a}} \end{array}
 \end{array} \quad (9)$$

To summarise, the physical fields and equations of motion reside in superfields

In order to use this information to extract a supersymmetric action principle, one needs an action containing the upper superfields, whose equations of motion contain the lower ones. The operation of going from fields to equations of motion looks like an exterior derivative in a fermionic direction. It indeed is, but in addition a projection has been performed, where 1-form parts have been projected away. This will be the rôle of the pure spinor.

Remember that a pure spinor  $\lambda$  is defined by

$$(\lambda\gamma^a\lambda) = 0, \tag{10}$$

so that the non-vanishing bilinears in  $\lambda$  are  $(\lambda\gamma^{ab}\lambda)$  and  $(\lambda\gamma^{abcde}\lambda)$ . (The precise statement is particular to  $D = 11$ . In  $D = 10$ , the remaining bilinear is a self-dual 5-form,  $(\lambda\gamma^{abcde}\lambda)$ .) Now, let us replace the fermionic frame form  $E^\alpha$  by  $\lambda^\alpha$ . For a  $p$ -form  $\omega$  pointing in the fermionic directions this simply means replacing

$$\omega = \frac{1}{p!} E^{\alpha_1} \wedge \dots \wedge E^{\alpha_p} \omega_{\alpha_p \dots \alpha_1} \tag{11}$$

by  $\frac{1}{p!} \lambda^{\alpha_p} \dots \lambda^{\alpha_1} \omega_{\alpha_1 \dots \alpha_p}$ . In ordinary superspace, taking an exterior derivative means mixing components with bosonic indices into the result, due to the presence of torsion:

$$(d\omega)_{\alpha_1 \dots \alpha_{p+1}} = (p+1)D_{(\alpha_1} \omega_{\alpha_2 \dots \alpha_{p+1})} + \binom{p+2}{2} T_{(\alpha_1 \alpha_2]}{}^a \omega_{|a| \alpha_3 \dots \alpha_{p+1})}, \tag{12}$$

where  $T_{\alpha\beta}{}^a = 2\gamma_{\alpha\beta}^a$ . It is not consistent to treat the fermionic directions only. However, the second term is projected away by the pure spinor constraint. So, the projection on certain modules performed by replacing the vielbein by the pure spinor allows for a consistent treatment of the components along fermionic directions alone.

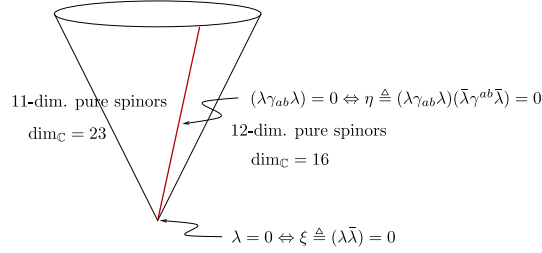
In this vein, a pure spinor superfield  $\Psi(x, \theta, \lambda)$ , with an expansion

$$\Psi(x, \theta, \lambda) = \psi(x, \theta) + \lambda^\alpha \psi_\alpha(x, \theta) + \frac{1}{2} \lambda^\alpha \lambda^\beta \psi_{\alpha\beta}(x, \theta) + \dots \tag{13}$$

provides a way of dealing with fermionic forms (of arbitrary degree) in a consistent manner. We will now make the correspondence between the supergravity vielbein and 3-form and this procedure more precise.

A scalar field  $\Psi(x, \theta, \lambda)$ , when expanded in a power series in  $\lambda$ , contains

$$1 \rightarrow \alpha \rightarrow \left( \begin{array}{|c|} \hline \square \\ \hline \oplus \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right) \rightarrow \left( \begin{array}{|c|} \hline \square \\ \hline \oplus \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right) \rightarrow \left( \begin{array}{|c|c|} \hline \square & \square \\ \hline \oplus & \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right) \rightarrow \dots \tag{14}$$



We recognise the modules of  $C_{\alpha\beta\gamma}$  and of the equations of motion. The cohomology of  $Q$ , as defined in eq. (2) gives the linearised equations of motion! A completely analogous statement holds for a field  $\Phi^a$  and the linearised supergeometry. In that case  $\Phi^a$  enjoys the extra gauge symmetry  $\Phi^a \approx \Phi^a + (\lambda\gamma^a\varrho)$  (which can also be understood using transformations corresponding to conventional constraints) [8].

This makes it clear how conventional superspace in a natural way leads to pure spinors. Both the fields and the modules implying the equations of motion can be interpreted as sitting in a pure spinor superfield (actually, both come in the same field). This opens for the possibility of going off-shell for such a field. The linearised equations of motion will be encoded as  $Q\Psi = 0$  or  $Q\Phi^a = 0$ .

Before turning to examining the implications of this, I would like to say some words about the pure spinor space. The pure spinor constraint only has solutions for complex  $\lambda$ , and the solution space turns out to be 23-dimensional. 9 out of the 11 constraints on the 32-dimensional spinor are independent.

There is a special 16-dimensional subspace of the pure spinor cône where not only  $(\lambda\gamma^a\lambda)$ , but also  $(\lambda\gamma^{ab}\lambda)$  vanishes. This is the space of 12-dimensional pure spinors. Here is a difference from  $D = 10$  where any monomial in  $\lambda$  consists of one irreducible module. While the only singular point in  $D = 10$  pure spinor space is the tip of the cône,  $\lambda = 0$ , there is a singular subspace in  $D = 11$ . This will be relevant later, when we consider operators on the pure spinor space.

The pure spinor superfields considered earlier are holomorphic in the complex variables  $\lambda^\alpha$ . This (and other issues) raises the question of how integration with respect to  $\lambda$  should be performed. By looking at the cohomology of the BRST operator  $Q$ , we will get a hint. We have already seen that the cohomology (at  $\lambda^3$  in the scalar field  $\Psi$ , corresponding to the superfield  $C_{\alpha\beta\gamma}$ , and at  $\lambda$  in  $\Phi^a$ , corresponding to the linearised field  $E_\mu^a$ ) contains the physical fields. There is clearly a gauge invariance, e.g.  $\delta\Psi = Q\Lambda$ . A careful examination shows that there is a cohomology in  $\Psi$  at  $\lambda^2$  containing gauge transformations (not only tensor gauge transformations, but also diffeomorphisms and local supersymmetry). But since  $\lambda$  has “wrong” statistics, this cohomology in  $\Psi$  will have opposite statistics compared to gauge parameters. These are ghost fields. Indeed, it turns out that the cohomology

encodes also the reducibility of the tensor gauge transformations/ghosts, all the way down to the scalar ghost-for-ghost-for-ghost, which sits as the  $\lambda$ - and  $\theta$ -independent part of  $\Psi$ . Corresponding statements are true of  $\Phi^a$ , but the gauge transformations encoded are only diffeomorphisms and local supersymmetry.

Let us for a moment specialise on the zero-mode cohomology, i.e., the cohomology of an  $x$ -independent field  $\Psi(\theta, \lambda)$ . Why? We have argued that  $Q\Psi = 0$  is the condition that enforces the equations of motion, which are some differential equations with respect to  $x$ . For zero-modes, they are automatically satisfied, and the cohomology problem turns into a purely algebraic problem. It can be solved by hand, or with computer assistance. A little thinking also tells us that if there are equations of motion imposed by cohomology on the fields at  $\lambda^p$ , these must be represented in the zero-mode cohomology at  $\lambda^{p+1}$ . In addition to the zero-mode cohomology giving physical fields and ghosts, there is in the cohomology of  $\Psi$  a complete “mirror” of fields, where the same (more generically, conjugate, but all  $so(11)$  modules are self-conjugate) modules occur at  $\lambda^p$  and  $\lambda^{7-p}$ . The cohomology at  $\lambda^4$  has the right properties to represent field equations or currents (as we have already argued earlier). The “wrong” statistics again forces an interpretation on us: they are antifields, in the Batalin–Vilkovisky (BV) [25] sense. All fields and ghosts and their respective antifields naturally occur as cohomology. A complete table of the zero-mode cohomology in  $\Psi$  is given below. The modules are given with their Dynkin label.

In the table, ghost numbers and dimensions have been assigned by demanding that those of the physical fields are correct. I would like to point at the “highest” cohomology, corresponding to the antifield for the ghost-for-ghost-for-ghost. This component of  $\Psi$ , itself a field with dimension  $-3$  and ghost number  $3$ , has dimension  $5$  and ghost number  $-4$ . Suppose we try to define integration as a kind of residue, by taking the component of an integrand in this cohomology. Such an integration would have dimension  $-8$  and ghost number  $-7$ . Consider a linearised action of the type

$$\kappa^2 S_0 \sim \int \Psi Q\Psi. \quad (15)$$

Together with integration over theta, the total dimension and ghost number of the component Lagrangian would be

$$\begin{aligned} \dim(\kappa^2 L_0) &= 2 \times (-3) - 8 + \frac{1}{2} \times 32 = 2, \\ \text{gh}\#(\kappa^2 L_0) &= 2 \times 3 + 1 - 7 = 0. \end{aligned} \quad (16)$$

This matches perfectly for a component Lagrangian.

One may therefore think that this solves the problem of finding a linearised action for the pure spinor superfield  $\Psi$ , whose equation of motion is  $Q\Psi = 0$  and reproduces the linearised supergravity multiplet, at least around flat

gh# =	3	2	1	0	-1	-2	-3	-4	-5
dim = -3	(00000)								
$-\frac{5}{2}$	•	•							
-2	•	(10000)	•						
$-\frac{3}{2}$	•	•	•	•					
-1	•	•	(01000) (10000)	•	•				
$-\frac{1}{2}$	•	•	(00001)	•	•	•			
0	•	•	•	(00000) (00100) (20000)	•	•	•		
$\frac{1}{2}$	•	•	•	(00001) (10001)	•	•	•	•	
1	•	•	•	•	•	•	•	•	•
$\frac{3}{2}$	•	•	•	•	(00001) (10001)	•	•	•	•
2	•	•	•	•	(00000) (00100) (20000)	•	•	•	•
$\frac{5}{2}$	•	•	•	•	•	(00001)	•	•	•
3	•	•	•	•	•	(01000) (10000)	•	•	•
$\frac{7}{2}$	•	•	•	•	•	•	•	•	•
4	•	•	•	•	•	•	(10000)	•	•
$\frac{9}{2}$	•	•	•	•	•	•	•	•	•
5	•	•	•	•	•	•	•	(00000)	•

space. This is not yet the case, but it is a big step on the way. The remaining problem lies in the observation that the integration is singular. A “residue” does not provide a non-singular measure when the power expansion is limited from one side. It is a bit like trying to define a residue for polynomials. This difficulty was resolved by the introduction of non-minimal variables [10]. In addition to  $\lambda$ , one also considers the pure spinor  $\bar{\lambda}$  and a fermionic spinor  $r^\alpha$  which is pure relative to  $\bar{\lambda}$ ,  $(\bar{\lambda}\gamma^a r) = 0$ . The BRST operator is changed into

$$Q = \lambda^\alpha D_\alpha + r_\alpha \frac{\partial}{\partial \bar{\lambda}_\alpha}. \quad (17)$$

This does not affect the cohomology. I will not go further into details about non-minimal variables and integration in this lecture.

I would now like to discuss the question of interactions. But first, a few words on the two fields,  $\Psi$  and  $\Phi^a$ . Each of them is capable of completely describing the linearised supergravity multiplet. An important difference is that while  $\Psi$  contains the “naked” 3-form potential and the associated ghosts,  $\Phi^a$  does not. It only contains the 3-form through its 4-form field



strength  $H = dC$ . A further observation is that the cohomology of  $\Phi^a$  (not presented in detail above), although having a kind of “mirror symmetry”, does not show a symmetry between fields and antifields. When one goes beyond equations of motion, the cohomology looks “too big”. Neither does this cohomology possess a singlet that can be related to a measure.

Since the component action contains a Chern–Simons term  $\sim C \wedge H \wedge H$ , it can never be constructed from  $\Phi^a$  alone. We must think of  $\Psi$  as the fundamental field and  $\Phi^a$  as a derived field. It therefore seems likely that there is some operator  $R^a$ , such that  $\Phi^a = R^a \Psi$ . Since cohomology should map to cohomology,  $R^a$  itself should commute with  $Q$  (modulo gauge transformation  $\delta \Phi^a = (\lambda \gamma^a \varrho)$ ). It is indeed possible to construct such an operator, with the correct quantum numbers. Here, I will not give the full form.

$$R^a = \eta^{-1} (\bar{\lambda} \gamma^{ab} \bar{\lambda}) \partial_b + \dots, \quad (18)$$

where the ellipsis denotes terms with  $r$ .  $\eta$  is the invariant vanishing on the 16-dimensional subspace of 12-dimensional pure spinors. This is again a difference from  $D = 10$ , where operators with negative ghost number typically diverge only at  $\lambda = 0$ . The operator  $R^a$  turns out to provide key input for the construction of interaction terms.

It turns out to be very fruitful to play with the fields  $\Psi$  and  $\Phi^a$ , and ask for possible 3-point couplings matching the counting of dimension and ghost number (no dimensionful constants should be included, unless one looks for higher-derivative interactions [24]). The extra gauge invariance for  $\Phi^a$  can be taken care of by demanding that it always sits in a combination  $(\lambda \gamma_{ab} \lambda) \Phi^b$ . Then the Fierz identity  $(\lambda \gamma_{ab} \lambda) (\gamma^b \lambda)^\alpha = 0$ , holding for a pure spinor, assures gauge invariance. Such a factor can also help to contract the vector indices on two (fermionic)  $\Phi$ 's. Simple counting shows that the combination

$$S_1 \sim \int \Psi (\lambda \gamma_{ab} \lambda) \Phi^a \Phi^b = \int (\lambda \gamma_{ab} \lambda) \Psi R^a \Psi R^b \Psi \quad (19)$$

is the only gauge invariant combination of  $\Psi$ 's and  $\Phi$ 's, without extra operators, that has the correct dimension and ghost number.

Could this be a good 3-point coupling? What are the principles in deciding which interaction terms are allowed? In order to answer these questions, we need to talk a little more about the BV formalism<sup>2</sup>. A good review, departing from classical field theory, is provided in ref. [26].

The BV formalism, in general, builds on a “doubling” of all fields, physical ones as well as ghosts, with their corresponding antifields, of opposite statistics. A fundamental structure, similar to a Poisson bracket, is provided by the antibracket, which in a component formalism is defined as

$$(A, B) = \int [dx] \left( A \frac{\overleftarrow{\delta}}{\delta \phi^A(x)} \frac{\overrightarrow{\delta}}{\delta \phi_A^*(x)} B - A \frac{\overleftarrow{\delta}}{\delta \phi_A^*(x)} \frac{\overrightarrow{\delta}}{\delta \phi^A(x)} B \right). \quad (20)$$

<sup>2</sup>In principle, one analyse interactions in terms of gauge invariance. But since both the action and the gauge transformations may get modifications, the BV framework turns out to be much more efficient, in that it deals with both issues at once.

Here,  $\phi^A$  denote fields (including ghosts) and  $\phi_A^*$  antifields. The action itself is the generator of “gauge transformations”, generated as  $\delta X = (S, X)$ , where  $(\cdot, \cdot)$  is the antibracket. The governing equation generalising  $Q^2 = 0$  is the BV master equation [25]

$$(S, S) = 0, \quad (21)$$

and this is the only consistency check needed when introducing interactions.

BRST cohomology is an inherently linear concept, and the BV formalism is the appropriate way to generalise it to non-linear (interacting) theories. Since we already know the the BRST cohomology of a pure spinor superfields provides both fields and antifields, there is no choice but to follow the BV procedure. The difference from a component formulation is that we are dealing with a single field  $\Psi$ , encoding all fields and antifields. For the pure spinor superfield  $\Psi$ , the antibracket takes the simple form [1]

$$(A, B) = \int A \overleftarrow{\frac{\delta}{\delta \Psi(Z)}} [dZ] \overrightarrow{\frac{\delta}{\delta \Psi(Z)}} B, \quad (22)$$

which I interpret as another sign that we are on the right track (the integral here is over all variables).

The full BV action for  $D = 10$  super-Yang–Mills (and its dimensional reductions) is the Chern–Simons-like action

$$S = \int [dZ] \text{Tr} \left( \frac{1}{2} \Psi Q \Psi + \frac{1}{3} \Psi^3 \right) \quad (23)$$

(implicit in refs. [7, 10, 5]). Note that there is only a 3-point coupling; the quartic interaction arises on elimination of “auxiliary fields”, notably the lowest component in the superfield  $A_\alpha(x, \theta)$ .

An analogous formulation exists for the Bagger–Lambert–Gustavsson and Aharony–Bergman–Jafferis–Maldacena models in  $D = 3$ . The simplification there is even more radical: The component actions contain 6-point couplings, but the pure spinor superfield actions only have minimal coupling (i.e., 3-point interactions) [17, 18].

But I would like to turn back to supergravity. The fact that the operator  $R^a$  commutes with  $Q$  (modulo gauge transformations) ensures that the interaction term proposed above

$$S_1 \propto \int [dZ] (\lambda \gamma_{ab} \lambda) \Psi R^a \Psi R^b \Psi \quad (24)$$

is a nontrivial deformation respecting the master equation. The factor  $(\lambda \gamma_{ab} \lambda)$

- ensures that dimension and ghost number are correct,
- guarantees the invariance under  $\Phi^a \approx \Phi^a + (\lambda \gamma^a \varrho)$ ,

- makes possible a contraction of  $\Phi^a$ 's.

Some terms have been checked explicitly (Chern–Simons term, coupling of diffeomorphism ghosts encoding the algebra of vector fields), so it is clear that this gives the 3-point couplings of  $D = 11$  supergravity.

One may expect that an expansion around flat space would be non-polynomial. This is however not the case. Checking the master equation to higher order in the field involves commutators of  $R^a$ 's. The  $R^a$ 's don't commute, but “almost”.

$$\frac{1}{2}(\lambda\gamma_{ab}\lambda)[R^a, R^b] = \frac{3}{2}\{Q, T\}, \quad (25)$$

where  $T = 8\eta^{-3}(\bar{\lambda}\gamma^{ab}\bar{\lambda})(\bar{\lambda}r)(rr)(\lambda\gamma_{ab}w)$ . The master equation is *exactly* satisfied by

$$S = \int [dZ] \left[ \frac{1}{2}\Psi Q\Psi + \frac{1}{6}(\lambda\gamma_{ab}\lambda)(1 - \frac{3}{2}T\Psi)\Psi R^a\Psi R^b\Psi \right]. \quad (26)$$

Note the similarity of the 3-point coupling ( $\propto \Psi\Phi\Phi$ ) to the Chern–Simons term (which it indeed contains). After a field redefinition  $\Psi = (1 + \frac{1}{2}T\tilde{\Psi})\tilde{\Psi}$ :

$$S = \int [dZ] \left[ \frac{1}{2}(1 + T\tilde{\Psi})\tilde{\Psi}Q\tilde{\Psi} + \frac{1}{6}(\lambda\gamma_{ab}\lambda)\tilde{\Psi}R^a\tilde{\Psi}R^b\tilde{\Psi} \right]. \quad (27)$$

I would like to stress that this is quite a remarkable property. It seems that the elimination of auxiliary fields will reintroduce the non-polynomial property of the component supergravity. There is of course a price for this simplicity. The geometric picture is lost, when the fields are expanded around a background (in our case, a flat one). Even if the action is exact to all orders, it is not clear how to find solutions that correspond to exact solutions in gravity or supergravity.

A few words on gauge fixing. In the BV formalism, it amounts to ordinary gauge fixing of the physical fields, as well as elimination of antifields. Covariant gauge fixing (Siegel gauge) amounts to demanding

$$b\Psi = 0,$$

where  $b$  is the composite  $b$ -ghost, satisfying  $[Q, b] = \square$ . The propagator then becomes  $b\square^{-1}$ . Unlike in component BV formalism, there is no need to introduce non-minimal fields (antighost, Nakanishi–Lautrup field); they are contained in  $\Psi$  (implicit in ref. [27]). The  $D = 11$   $b$ -ghost has been constructed [28], and takes the form

$$b = \frac{1}{2}\eta^{-1}(\bar{\lambda}\gamma_{ab}\bar{\lambda})(\lambda\gamma^{ab}\gamma^i D)\partial_i + \dots$$

Some conclusions and problems:

- The framework described resolves the issue of classical supersymmetric actions for maximally supersymmetric theories.
- The interaction terms are generically much simpler and of lower order than in a component language; for supergravity to the extent that the action becomes polynomial.
- Presumably, the formalism may be efficient for calculating quantum amplitudes. Need to establish connection to “superparticle” prescription. Finiteness of BLG? Of  $N = 8$  supergravity? Regularisation is needed in path integrals, due to negative powers of  $\eta$ .
- How is U-duality realised? Models connected to generalised geometry, with enlarged structure groups, may possibly provide generalised models of gravity?
- Geometry? Background invariance? The polynomial property should be better understood.

## References

- [1] M. Cederwall, “Towards a manifestly supersymmetric action for  $D=11$  supergravity”, J. High Energy Phys. **1001** (2010) 117 [arxiv0912.1814], “ $D=11$  supergravity with manifest supersymmetry”, Mod. Phys. Lett. **A25** (2010) 3201 [arxiv1001.0112].
- [2] B.E.W. Nilsson, “Pure spinors as auxiliary fields in the ten-dimensional supersymmetric Yang–Mills theory”, Class. Quantum Grav. **3** (1986) L 41.
- [3] P.S. Howe, “Pure spinor lines in superspace and ten-dimensional supersymmetric theories”, Phys. Lett. **B258** (1991) 141.
- [4] N. Berkovits, “Super-Poincaré covariant quantization of the superstring”, J. High Energy Phys. **0004** (2000) 018 [hep-th/0001035].
- [5] M. Cederwall, B.E.W. Nilsson and D. Tsimpis, “The structure of maximally supersymmetric super-Yang–Mills theory—constraining higher order corrections”, J. High Energy Phys. **0106** (2001) 034 [hep-th/0102009].
- [6] M. Cederwall, B.E.W. Nilsson and D. Tsimpis, “ $D=10$  super-Yang–Mills at  $\beta O(\alpha'^2)$ ”, J. High Energy Phys. **0107** (2001) 042 [hep-th/0104236].
- [7] N. Berkovits, “Covariant quantization of the superparticle using pure spinors”, J. High Energy Phys. **0109** (2001) 016 [hep-th/0105050].
- [8] M. Cederwall, B.E.W. Nilsson and D. Tsimpis, “Spinorial cohomology and maximally supersymmetric theories”, J. High Energy Phys. **0202** (2002) 009 [hep-th/0110069]; M. Cederwall, “Superspace methods in string theory, supergravity and gauge theory”, Lectures at the XXXVII Winter School in Theoretical Physics “New Developments in Fundamental Interactions Theories”, Karpacz, Poland, Feb. 6-15, 2001, [hep-th/0105176].
- [9] M. Movshev and A. Schwarz, “On maximally supersymmetric Yang–Mills theories”, Nucl. Phys. **B681** (2004) 324 [hep-th/0311132].
- [10] N. Berkovits, “Pure spinor formalism as an  $N=2$  topological string”, J. High Energy Phys. **0510** (2005) 089 [hep-th/0509120].
- [11] M. Cederwall and B.E.W. Nilsson, “Pure spinors and  $D=6$  super-Yang–Mills”, [arxiv0801.1428].
- [12] M. Cederwall, U. Gran, M. Nielsen and B.E.W. Nilsson, “Manifestly supersymmetric  $M$ -theory”, J. High Energy Phys. **0010** (2000) 041 [hep-th/0007035]; “Generalised 11-dimensional supergravity”, [hep-th/0010042].

- 
- [13] M. Cederwall, U. Gran, B.E.W. Nilsson and D. Tsimpis, “*Supersymmetric corrections to eleven-dimensional supergravity*”, J. High Energy Phys. **0505** (2005) 052 [hep-th/0409107].
- [14] P.S. Howe and D. Tsimpis, “*On higher order corrections in M theory*”, J. High Energy Phys. **0309** (2003) 038 [hep-th/0305129].
- [15] P.S. Howe, “*Pure spinors, function superspaces and supergravity theories in ten and eleven dimensions*”, Phys. Lett. **B273** (1991) 90.
- [16] P. Fré and P.A. Grassi, “*Pure spinor formalism for  $OSp(N|4)$  backgrounds*”, [arxiv0807.0044].
- [17] M. Cederwall, “ *$N=8$  superfield formulation of the Bagger–Lambert–Gustavsson model*”, J. High Energy Phys. **0809** (2008) 116 [arxiv0808.3242].
- [18] M. Cederwall, “*Superfield actions for  $N=8$  and  $N=6$  conformal theories in three dimensions*”, J. High Energy Phys. **0810** (2008) 070 [arxiv0809.0318].
- [19] R. Marnelius and M. Ögren, “*Symmetric inner products for physical states in BRST quantization*”, Nucl. Phys. **B351** (1991) 474.
- [20] N. Berkovits, “*ICTP lectures on covariant quantization of the superstring*”, proceedings of the ICTP Spring School on Superstrings and Related Matters, Trieste, Italy, 2002 [hep-th/0209059].
- [21] E. Cremmer, B. Julia and J. Scherk, “*Supergravity theory in eleven-dimensions*”, Phys. Lett. **B76** (1978) 409.
- [22] L. Brink and P. Howe, “*Eleven-dimensional supergravity on the mass-shell in superspace*”, Phys. Lett. **B91** (1980) 384; E. Cremmer and S. Ferrara, “*Formulation of eleven-dimensional supergravity in superspace*”, Phys. Lett. **B91** (1980) 61.
- [23] S.J. Gates, K.S. Stelle and P.C. West, “*Algebraic origins of superspace constraints in supergravity*”, Nucl. Phys. **B169** (1980) 347; S.J. Gates and W. Siegel, “*Understanding constraints in superspace formulation of supergravity*”, Nucl. Phys. **B163** (1980) 519.
- [24] M. Cederwall, U. Gran, M. Nielsen and B.E.W. Nilsson, “*Manifestly supersymmetric M-theory*”, J. High Energy Phys. **0010** (2000) 041 [hep-th/0007035]; “*Generalised 11-dimensional supergravity*”, hep-th/0010042.
- [25] I.A. Batalin and G.I. Vilkovisky, “*Gauge algebra and quantization*”, Phys. Lett. **B102** (1981) 27.
- [26] A. Fuster, M. Henneaux and A. Maas, “*BRST-antifield quantization: a short review*”, hep-th/0506098.
- [27] Y. Aisaka and N. Berkovits, “*Pure spinor vertex operators in Siegel gauge and loop amplitude regularization*”, J. High Energy Phys. **0907** (2009) 062 [arxiv0903.3443].
- [28] Y. Aisaka, N. Berkovits and M. Cederwall, unpublished.