Quantum Integrability and Entanglement Generators *

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Abstract

Dynamical algebra notion of quantum degrees of freedom provides a useful and natural definition of quantum integrable and nonintegrable systems. We have argued that a quantum dynamical system generates generalized entanglement by internal dynamics if and only if it is quantum non-integrable.

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Our goal will be to study the importance of the definition of independent degrees of freedom (IDF) for the relation between entanglement and quantum integrability for a quantum dynamical system in general. We shall see that in quantum mechanics the choice of degrees of freedom dictated by the dynamical structure of the system that is by the dynamical algebra and its particular subalgebras should represent also an appropriate choice of IDF for an objective and generalized treatment of entanglement. Discussion of the systems dynamical group will lead to a notion of degrees of freedom and the discussion of dynamical symmetry to the notion of quantum non-integrability and entanglement generating systems.

In the next section we shall first recapitulate general definitions of independent degrees of freedom, quantum integrability and the generalized entanglement. We then establish the relation between the quantum integrability and generalized entanglement and discuss, in section 3, some examples. Dynamical algebraic definition of IDF and quantum integrability have been introduced in references [1, 2, 3]. There is no generally accepted notion of genuinely quantum integrability [4]. The definition of what is a quantum chaotic system is even less unique [5]. The most common approach, at least for lattice spin systems, is based on the generalization of the notion of thermodynamical integrability of classical spin systems[6], and is different from the one accepted here. According to the thermodynamic integrability a quantum system is called integrable if it is exactly solvable

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by application of the generalized Bethe ansatz or by the quantum inverse scattering method [4]. A quantum system is nonintegrable if it has not been integrated by such methods. In what sense a quantum nonintegrable system can be considered as quantum chaotic is a matter of a debate [7]. Some quantum systems of finite number of spins whose thermodynamical limit is quantum nonintegrable, show the same spectral properties as the systems obtained by quantization of classically chaotic systems, and, furthermore, display the mixing properties that lead to expected equilibrium and non-equilibrium thermodynamical behavior [7]. The dynamics of bipartite and multipartite entanglement in such quantum chaotic systems has been studied and compared with the entanglement dynamics in quantum integrable systems [8],[9],[11] The notion of quantum integrability and nonintegrability understood in the thermodynamic sense is very different from the notion of dynamical symmetry and quantum integrability as was introduced in [1, 2, 3] and as it shall be used here.

The definition of generalized entanglement adopted here was presented in [12, 13, 14], and related definitions appear for example in [15],[16]. Here presented view of the general relation between quantum integrability, dynamics of classical approximations of quantum systems and dynamics of the generalized entanglement has not been discussed before.

1. General definitions

1.1. Dynamical algebra framework

Any quantum system with an N-dimensional Hilbert space has $N-1$ kinematical degrees of freedom. Its group of canonical transformations, i.e. the kinematical group, is $U(N)$ so that any Hamiltonian, i.e. Hermitian operator, can be diagonalized using some of the $U(N)$ transformations, leading to $N-1$ formal integrals of motion and formal integrability. Evolution of the quantum system is equivalent to a linear symplectic flow on a symplectic manifold, which is completely integrable in the sense of classical Hamiltonian systems (please see [17] and the references therein). All pure states can be connected by some unitary transformation so that all pure states are in fact $U(N)$ generalized coherent states. The assumption that any hermitian operator represents a measurable quantity, an observable, is actually an assumption concerning physically possible interactions with the environment, and is not justified in many cases such as: systems of identical particles, presence of symmetries, relativistic locality etc....

Dynamical algebra of relevant observables

The kinematical notion of degrees of freedom is rather formal to be physically relevant. A particular physical system is specified, and thus distinguished from an abstract general framework, by describing what can be measured on it, i.e. by specifying the set of observables, and by expressing the interactions within the system in terms of the observables. In other words, the class of physically relevant observables should be described and the evolution should be expressed in terms of these observables. Structure of the set of observables is fixed by their algebraic relations. In quantum mechanics operators representing the physical quantities pertinent to the given system are required to realize the corresponding algebra. The algebraic relations between the observables also fix the relevant Hilbert space of the system as the space of an irreducible representation. The algebra defined in this way is called the systems dynamical algebra. Thus, a quantum system has fixed dynamical algebra. Description of a quantum system amounts to the specification of its dynamical algebra g, its state space and the Hamiltonian which is an expression (possibly nonlinear) in terms of operators belonging to g.

In what follows we shall consider a quantum system $(H, g, \mathcal{H})$ with a Hilbert space H which is an irrep space of the dynamical (Lie) algebra g and the Hamiltonian $\mathcal{H}$. The dynamical algebra g will always be a semi-simple Lie algebra, with rank l and dimension n.

1.2. Dynamical degrees of freedom

Dynamical degrees of freedom are fixed by the full description of the system, and are defined using the dynamical algebra. The dynamical algebra g has γ different chains of subalgebras: $g \supset g^l_{s^l} \supset g^l_{s^l-1} \cdots \supset g^l_1$, $l = 1, 2, \ldots \gamma$. Casimir operators of g and all the algebras in (any of) the subalgebra chain form the relevant complete set of commuting operators (CSCO) $Q_j, j = 1, 2 \ldots d$. There is $d = l + (n-l)/2$ of these, independently of the subalgebra chain. Some of these Casimir operators are fully degenerate in the sense that they are represented by scalar operators: $Q_i|\psi>= c_i|\psi>$ for every $|\psi>\in H$. The number of non-fully degenerate operators in CSCO is $m \leq (n-l)/2$ is chain independent but might depend on the particular irrep i.e. on the system's Hilbert space, and defines the number of IDF. The non-fully degenerate Casimir operators in a particular chain are the operators that define m IDF. The quantum system is fully specified only when 1^o its Hilbert space; 2^o the set of m operators representing IDF and 3^o the Hamiltonian, which is a possibly nonlinear expression in terms of the dynamical algebra generators, are given.

If the dynamical algebra g of a quantum system C can be represented as a direct sum of dynamical algebras of two systems A and B, that is $g^C = g^A \oplus g^B$, then the tensor product of irreps of G^A and G^B is an irrep space of G^C, that is $H^C = H^A \otimes H^B$. If $l_{A,B}$ and $n_{A,B}$ are the ranks and dimensions of g^A and g^B, then in general the number of IDF of C is $M_C = M_A + M_B$. Thus, in the case $g^C = g^A \oplus g^B$ the system C can be represented as a union of two systems and the number of IDF is additive. If the dynamical algebra g of the system is semisimple then it can be uniquely expressed as a direct sum of mutually commutative and orthogonal simple algebras: $g = \oplus_k g_k$ and the Hilbert space which is an irrep space of g factors as $H = \otimes_k H_k$. Thus, in the case of semisimple dynamical algebra the number of IDF is additive, but the number of IDF in all the factor systems with g_k dynamical algebras need not be unity for each g_k. An example of the system when this is the case is given by a system of qubits, and shall be treated in some detail later.

However, a dynamical algebra g need not be representable as a product of dynamical Lie algebras with the number of IDF equal to one (as for example if $g = su(3)$ or if A and B are independent fermions or bosons) even if the number of IDF of g is larger then one.

1.3. Dynamical symmetry and quantum integrability

$(H, g, \mathcal{H})$ has the corresponding Lie group G as the dynamical symmetry if the Hamiltonian $\mathcal{H}$ can be expressed in terms of the CSCO of a particular subgroup chain used to define the IDF. In this case the system has a symmetry of the subgroup chain. In particular H commutes with m non-fully degenerate operators that define the IDF.

A system $(H, g, \mathcal{H})$ is quantum integrable by definition if the Hamiltonian $\mathcal{H}$ commutes with m operators that define the IDF.

Quantum integrability is defined in analogy with complete integrability in the case of classical Hamiltonian systems. In classical Hamiltonian mechanics, an m degree of freedom system is completely integrable if there is m, functionally independent integrals of motion (including the Hamilton function) in involution. The requirement that the integrals are functionally independent is taken over into the quantum mechanical definition by requiring that the Hamiltonian commutes with operators representing the IDF. Quantum Hamiltonian systems which do not satisfy the definition of quantum integrability are called quantum nonintegrable. It should be stressed that the qualitative properties of the state dynamics with quantum integrable and quantum non-integrable Hamiltonians are the same. From the point of view of the Hamiltonian dynamical systems theory the state orbits are in either case regular that is periodic or quasi-periodic. Quantum non-integrable systems do not generate chaotic orbits in the system's state space (please see for example [17]). Nevertheless, dynamical properties of orbits of the classical models (please see the next subsection) corresponding to the quantum integrable or non-integrable systems are quite different, and chaotic orbits do occur in the classical model of quantum non-integrable systems with more than one IDF.

It should be noticed that quantum systems with one degree of freedom, unlike the one freedom classical Hamiltonian systems, need not be quantum integrable, for example if the Hamiltonian is a nonlinear expression of the algebra generators.

1.4. g-coherent states

Total level of quantum fluctuations in a pure state $|\psi>$ is defined as

$$\Delta(\psi) = \sum_{i}^{n} <\psi|L_i^2|\psi> - <\psi|L_i|\psi>^2, \tag{1}$$

where the sum is taken over an orthonormal bases of the dynamical algebra g. It make sense to consider the quantity $\Delta(\psi)$ as a measure of quantumness of the state ψ. Physical motivation for the definition of the generalized g-coherent states is that they minimize $\Delta(\psi)$. This is one of the important properties of the Glauber coherent states of the harmonic oscillator i.e. of the Heisenberg-Weil H_4 algebra that is generalized by the g-coherent states with arbitrary g. There are several generalizations of Glauber, i.e. H_4 coherent states. Perelomov [18] and Gilmore [19] independently introduced two different generalizations based on the group-theoretical structure of the H_4 coherent states. The essential ideas of both approaches are the same, the differences being in the class of Lie groups, and the corresponding available tools, and in the choice a reference state. In both approaches, the set of g-coherent states depends, besides the algebra g, also on the particular Hilbert space H^Λ carrying the irrep Λ of g and on the choice of an, in principal (Perelomov), arbitrary referencee state, denoted $|\psi_0 >$. The subgroup S_{ψ_0} of G which leaves the ray corresponding to the state $|\psi_0 >$ invariant is called the stability subgroup of $|\psi_0 >$: $h|\psi_0 >= |\psi_0 > \exp\ i\chi(h), h \in S_{\psi_0}$. Then, for every $g \in G$ there is a unique decomposition into the product of two elements, one from S_{ψ_0} and one from the coset G/S_{ψ_0} so that $g|\psi_0 >= \Omega|\psi_0 > \exp\ i\chi(h)$. The states of the form $|\Lambda, \Omega >= \Omega|\psi_0 >$ for all $g \in G$ are the g coherent states. Thus, geometrically the set of g coherent states form a manifold with well defined Riemannian and symplectic structure.

Classical model and semi-classical dynamics

Classical Hamiltonian dynamical system on the manifold $G/S_{|\psi_0>}$ given by the Hamiltonian function $\mathcal{H}(\alpha) =< \alpha|\mathcal{H}|\alpha >$ is called the classical model of the quantum dynamical system $(H, g, \mathcal{H})$. Classical limit of the quantum system is obtained from the classical model in the limit when some relevant parameter approaches zero. If the Hamiltonian $\mathcal{H}$ is a linear expression of the dynamical group generators then the quantum system, its classical model and its classical limit have the same dynamics. The classical model of a quantum nonintegrable system is chaotic in the sense of classical Hamiltonian dynamical systems. Dynamics of classical models of quantum nonintegrable systems have been studied for various examples in [1, 2, 3]. Relation between dynamics of entanglement and the dynamics of classical models for quantum nonintegrable pair of qubits was studied in [20].

1.5. Generalized entanglement

Consider a system C such that its dynamical group G^C can be factored as a direct product $G^C = G^A \otimes G^B$, and its Hilbert space H^C written as the tensor product of the irreps $H^C = H^A \otimes H^B$. We have seen that such a system C can be viewed as a union of systems A and B. A pure state ψ_C of C is entangled by the standard definition if the reduced states $\rho_{A,B} = Tr_{B,A}[|\psi_C >< \psi_C|]$ are mixtures i.e. are not pure state projectors. In this case the subsystems have no definite properties.

g-coherent states of the system C with $G^C = G^A \otimes G^B$ are products of g^A and g^B coherent states, by the construction of coherent states with the referent state $\psi_0^C = |\psi_0^A > \otimes|\psi_0^B >$ and are of the form $|\alpha^A >$

$\otimes|\alpha^B>= G^A|\psi_0^A> \otimes G^B|\psi_0^B>$. Reduced states $\rho_{A,B}$ of the coherent state $|\alpha^A> \otimes|\alpha^B>$ are pure and are coherent states of A and of B respectively. The coherent states of G^C are disentangled, and the reduced state of the coherent state are also coherent, and thus disentangled for the component algebras. Thus, the set of states with zero entanglement and the set of coherent states can be consistently identified. In this sense the noncoherent states do posses some entanglement in the generalized sense. If A and B are systems with only one IDF each, the previous definition assumes that the noncoherent states of A and B are entangled in the generalized sense. These states $|\psi^A>, |\psi^B>$ of systems A and B with number of IDF equal to unity do have nonminimal quantumness $\Delta(\psi)$, and violate Bell inequality for some set of observables [15].

In the considered case the quantumness $\Delta(\psi^C)$ is in general larger than minimal, the minimum being achieved by states which are products of G^A and G^B coherent states. The quantumness of the state $|\psi^C>$ is here manifested in one of the two modes: a) by quantum correlations between different IDF, which is traditionally identified with entanglement, or by b) quantumness of states of systems with unit number of IDF. The definition of generalized entanglement assumes that nonminimal quantumness of noncoherent states of systems with one IDF is equivalent to the generalized entanglement. In either case *a*) or *b*) some Bell inequality for a convenient choice of observables is violated by a superposition of g-coherent states, that is by generalized entangled states.

Previous discussion in the case when the dynamical group satisfies $G = G^A \otimes G^B$ is generalized by definition to the general case of the systems with dynamical groups G such that the decomposition $G = G^A \times G^B$ does not exist. Although the system with such dynamical algebra might have more than one IDF it can not be considered as a union of systems with smaller number of IDF. Nevertheless, the g-coherent states are defined and constructed as in the general case. The quantumness $\Delta(\psi)$ is minimal for such coherent states and larger than minimal otherwise. States which are not g-coherent have the quantumness larger than minimal and are by definition generalized entangled or g-entangled states. Quantumness of the state $|\psi>$: $\Delta(\psi)$, normalized so that it is zero for the g-coherent states can be used as a measure of g-entanglement. It was shown in [14] that it is related to the Mayer-Wallach Q-measure of multi-partite entanglement in the standard case.

Identification of g-coherent states with g-disentangled states in the case when the dynamical group does not satisfy $G = G^A \otimes G^B$ should not be questionable. Whether a state should be considered as g-entangled whenever it is not g-coherent is a deep question with no general agrement as to the answer [15]. Following [12, 13, 14] we adopt the identification of g-entanglement with g-noncoherence. This reduces to the standard definition in the case when A and B have no entangled states and $G = G^A \otimes G^B$.

If this definition of g-entanglement is adopted than quantum integrability and g-entanglement are clearly related as is explained in the next subsection.

1.6. Entanglement generator and integrability

If a quantum system $(g, H, \mathcal{H})$ is integrable then the dynamics commutes with IDF. The set of coherent states is then dynamically invariant, and consequently the system does not evolve an initially g-nonentangled into an q-entangled state. On the other hand if $(g, H, \mathcal{H})$ is nonintegrable the set of g-coherent e.i. g-nonentangled states is not dynamically invariant and thus $(g, H, \mathcal{H})$ generates entanglement. This properties provide an understanding of the relation between dynamical integrability and generalized entanglement in quantum mechanics and is the main conclusion of our discussion.

A quantum engineer who wants to generate entangled states cheaply , i.e. by the system's natural dynamics, should use a nonintegrable system for this purpose.

2. Examples

2.1. von-Neumann case: $u(N)$ dynamical algebra

The quantum system is described by N dimensional Hilbert space H^N and the dynamical algebra $u(N)$, which means that every hermitian operator on H^N has physical interpretation as a measurable quantity. Due to the normalization and global phase invariance the state space of the system is CP^{N-1} which is topologically like S^{2N-1}/S^1, and represents a $2(N-1)$ dimensional manifold with Riemannian and symplectic structure. Geometrically, it should be natural to associate $N-1$ IDF with this system. The same number of IDF follows from $u(N)$ dynamical algebra. The Hilbert space is the fully symmetric irrep space of $u(N)$ with the highest weight: $\Lambda = (1, 0, \ldots 0)$. The basis can be labeled by the following chain of subalgebras: $u(N) \supset u(N-1) \cdots \supset u(1)$ with the corresponding Casimir operators $C_i^{u(k)}$, $i = 1, 2 \ldots k$, $k = 1, 2 \ldots N$ determine the irrep $\Lambda = (1, 0, \ldots 0)$. The $N-1$ non-fully degenerate operators are $C_i^{u(k)}$, $i = 1, 2 \ldots k$, $k = 1, 2 \ldots N-1$ and label the basis $|i >= |0, 0, \ldots i, \ldots 0 >, i = 0, 1, 2, \ldots N-1$. Explicitly:$C_k^{u(k)} |i >= \Theta(k-(N-i))|i >$, and $\Theta(i)$ is the Heaviside function on $i = 1, 2 \ldots N-1$. Thus there is $N-1$ IDF, the same as the number of kinematical DF.

Any Hamiltonian can be diagonalized by an $U(N)$ transformation and expressed as a combination of the Casimir operators. Thus any system with $u(N)$ dynamical algebra is quantum integrable. The classical model for any quantum system with $u(N)$ dynamical algebra is also completely integrable when considered as a classical Hamiltonian system.

Elementary excitation operators are given by: $E_{i0}|\psi_0 >= |i >, i = 1, 2, \ldots N-1$ where $|\psi_0 >$ is the lowest weight vector of the $\Lambda = (1, 0, \ldots 0)$ representation, and $U(N)$ coherent states are obtained as $|\alpha >= exp(\sum \alpha_i E_{i1} - h.c)|0 >$. Coherent states are parameterized by the coset space $U(N)/U(N-$

$1) \otimes U(1)$ which is isomorphic to CP^{N-1}. We see that all states are $U(N)$ coherent states. Thus, all states are equally and minimally quantum. The $N-1$ degrees of freedom are disentangled in any state.

It should be noticed that since any state is $u(N)$ coherent state the dynamics of the quantum system on CP^{N-1} and its classical model with the Hamiltonian function $< \mathcal{H} >$ on the phase space $U(N)/U(N-1) \otimes U(1)$ are identical (and integrable) for any Hamiltonian (please see for example [17]).

A special case of the systems with $su(N)$ dynamical group is a qubit with $su(2)$ dynamical algebra and the Hilbert space with two complex dimensions. The number of degrees of freedom of the qubit is one, and all states, like in the general $u(N)$ case, are coherent and equally and minimally quantum.

Systems with $su(2)$ dynamical algebra but with the Hilbert space with $dim > 2$ are treated next.

2.2. Entanglement and quantum nonintegrability in a system with one IDF: $su(2)$ dynamical algebra with $dimH = 2j+1 > 2$

The two Casimir operators in the subalgebra chain: $su(2) \supset u(1)$ are J^2 and J_0. The system has only one IDF, given by the only one non-fully degenerate operator J_0. Hamiltonian which is a linear expression of the $SU(2)$ generators is quantum integrable according to the definition (with the proper choice of the quantization axes). A system with a Hamiltonian that is a nonlinear expression of the generators is quantum nonintegrable, and as we shall see generates g-entanglement.

The $SU(2)$ coherent states are: $|\alpha >= \exp((\alpha J_+ - h.c)|0 >$ where $|0 >$ is the unique lowest weight vector in the representation H^{2j+1} and J_+ is the corresponding raising operator. States which are not coherent are more quantum in the sense that they have larger Δ than the coherent states. According to the accepted definition such states are g-entangled.

Consider the Hamiltonian (2) modelling the dynamics of two-mode Bose-Hubbard model with fixed number of particles $N = 2j$. The Hamiltonian is expressed in terms of the $su(2)$ generators and is integrable when $\mu = 0$ and nonintegrable when $\mu \neq 0$.

$$\mathcal{H} = \omega_z J_z - 2\omega_x J_x + \mu J_z^2, \tag{2}$$

If the Hamiltonian is a linear expression in terms of the $su(2)$ generators, i.e. an element of the $su(2)$ algebra, then the set of coherent states is dynamically invariant. The linear combination of generators can be considered as the single Casimir operator of $u(1)$ that defines the single IDF. The linear system is then quantum integrable. On the other hand, when the Hamiltonian is a nonlinear expression of the $su(2)$ generators the states with different levels of quantumness are not dynamically isolated. The system is quantum nonintegrable and numerical tests show that the nonintegrable Hamiltonian generates g-entanglement in the systems with one IDF.

2.3. Coupled qubits: $su^1(2) \oplus su^2(2)$ dynamical algebra

Consider a pair of spins with the Hilbert space $H = H^2 \otimes H^2$ and the Hamiltonian

$$H = (1-\mu)(J_z^1 + J_z^2) + \mu_1 J_x^1 J_x^2 + \mu J_z^1 J_z^2, \tag{3}$$

where $\mu \neq 1$.

The dynamical group of the system is $SU^1(2) \otimes SU^2(2)$. The subgroup chain $\alpha: SU^1(2) \otimes SU^2(2) \supset SO^1(2) \otimes SO^2(2)$ gives two IDF and the Casimir operators of the subgroups J_z^1 and J_z^2 are the observables corresponding to the two IDF.

When $\mu = 0$ the system is quantum integrable with respect to the considered IDF. The Hamiltonian commutes with J_z^1 and J_z^2. If $\mu_1 \neq 0$ and $\mu \neq 1$ the system is quantum nonintegrable. As was already pointed out, the orbits in the Hilbert space of the quantum integrable and nonintegrable cases belong in the same class from the point of view of the qualitative theory of dynamical systems, i.e. they are regular orbits.

Because of the definition of the dynamical group as $SU^1(2) \otimes SU^2(2)$ the system is considered as composed of two spins. Coherent states are products of the coherent states of each of the spins and are thus disentangled. If $\mu_1 = 0$ the system is quantum integrable with respect to α subgroup chain and the $SU^1(2) \otimes SU^2(2)$ coherent i.e. disentangled states are dynamically invariant. Such Hamiltonian does not generate entanglement despite the interaction $\mu J_z^1 J_z^2$ between the two qubits. If $\mu \neq 0$ the system is not quantum integrable and the sets of coherent i.e g disentangled and noncoherent i.e. g-entangled states are not dynamically invariant. The system generates entanglement between the $SO^1(2) \otimes SO^2(2)$ dynamical degrees of freedom.

3. Summary

We have used the dynamical algebra definition of independent degrees of freedom in order to establish a general relation between quantum integrability or nonintegrability and the dynamics of the generalized entanglement (g-entanglement). Quantum system is integrable by definition if the operators corresponding to IDF commute with the Hamiltonian. Minimal level of total quantum fluctuations is a property characteristic of the dynamical algebra generalized coherent states. States with non-minimal quantum fluctuations are here identified (following [12, 13, 14]) with the g-entangled states. With this identification, both sets of g-disentangled and g-entangled states are dynamically invariant for the quantum integrable systems. On the other hand, an orbit of the quantum nonintegrable system goes through states with zero and nonzero g-entanglement. Quantum nonintegrable systems generate g- entanglement by the internal dynamics, while quantum integrable systems can be in a g-entangled state only due to interactions with external systems.

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