New analytic solutions in SFT*

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Abstract

This is short review of tachyon condensation and open string field theory. After a brief introduction to open string theory, the SFT action is introduced and illustrated. Next comes tachyon condensation in the level truncation approach, which introduces the main topic: the description of the analytic solution and the proof of the first two conjectures by Sen. Finally we present a new proposal to construct analytic lump solutions in SFT and prove third Sen's conjecture.

1. Introduction

Tachyon condensation is a pervasive phenomenon in physics. Whenever a field theory has a potential with a local maximum, surrounded by (possibly a continuum of) local minima, quantization around the maximum brings about the appearance of an unphysical particle with negative square mass, the tachyon. The tachyon is simply the manifestation of the instability of the vacuum where the theory is quantized. Any tiny disturbance takes the system to a more stable configuration based on a local minimum (we say the tachyon has condensed). This is, for instance, the typical situation of the spontaneous breakdown of a symmetry. The subject of this review is tachyon condensation in a system of infinite many degrees of freedom, as described by string field theory (SFT). The motivations underlying the study of this system are both theoretical and applicative, and stem from the basic role of D–branes in the framework of string theory.

D–branes mean open strings: open strings (unlike closed strings) do not exist as autonomous entities but only when their endpoints can lie on D–branes (which may also be space-filling). On the other hand D–branes

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do not have an autonomous existence either: they are a geometrical abstraction representing the dynamics of the open strings attached to them. Studying the dynamics of open strings is therefore of utmost importance and tachyon condensation is basic in this respect. An example will suffice. A phenomenon like inflation can be described by the attractive potential between a D–brane and an anti–D–brane, at least as long as the two branes are far apart. However, when their distance becomes smaller than the string scale (after inflation has terminated) the string spectrum develops tachyons and the natural evolution of the system is represented by tachyon condensation.

In these lectures I will discuss bosonic open string field theory. Purely bosonic string theory is, of course, by itself insufficient, if anything because its spectrum does not contain fermions. However open string field theory is a simplified playground with respect to the corresponding superstring field theory versions. Exploiting the relative simplicity of the bosonic theory it has been possible in the last ten years to make significant progress and, then, export some of the results to the superstring relatives. Therefore our purpose here will be the description of tachyon condensation and related phenomena in the framework of Witten's Open String Field Theory [1], and the guideline for all these recent developments is represented by A.Sen's conjectures, [2, 3]. The latter can be summarized as follows. Bosonic open string theory in D=26 dimensions is quantized on an unstable vacuum, an instability which manifests itself through the appearance of the open string tachyon. The effective tachyonic potential has, beside the local maximum where the theory is quantized, a local minimum. Sen's conjectures concern the nature of the theory around this local minimum. First of all, the energy density difference between the maximum and the minimum should exactly compensate for the D25–brane tension characterizing the unstable vacuum (first conjecture): this is a condition for the (relative) stability of the theory at the minimum. Therefore the theory around the minimum should not contain any quantum fluctuation pertaining to the original (unstable) open string theory (second conjecture). The minimum should therefore correspond to an entirely new theory, which can only be the bosonic closed string theory. If so, in the new theory one should be able to find in particular all the classical solutions characteristic of closed string theory, the D25–brane as well as all the solitonic solutions representing lower dimensional D–branes (third conjecture).

The evidence in favor of these conjectures has been accumulating over the years although not with a uniform degree of accuracy and reliability, until the first two conjectures were rigorously proved, [7, 8]: an explicit analytic (non–perturbative) SFT solution was produced, which links the initial vacuum to the final one and it was shown that this vacuum does not contain perturbative open string modes. As for the third conjecture a recent proposal has been put forward recently, [20], as to how to construct analytic lump solutions. It is the purpose of this review to illustrate these results in turn.

subject of SFT and tachyon condensation, which would take an article the size of a book. There are already several reviews the reader can consult

[11, 12, 13, 10, 14], which cover different aspects and different subjects. My aim here is to give a general survey of the search for analytic solutions. But, before, I would like to mention a few important topics are that not considered in these lectures, but belongs in a natural development of the material covered here.

The D25–brane and its lower dimensional companions are unstable, because there is no conserved charge (like in the corresponding supersymmetric theories) associated to them. Therefore SFT must contain also time–dependent solutions that describe their decay. This issue has been discussed in [4, 5] and approximate solutions have been found in SFT, but uncontroversial analytic solutions are still lacking.

Finally, a very far–reaching consequence of Sen's conjectures has so far remained rather implicit in the literature. It is evident that if the three conjectures are true and the new vacuum is the closed string vacuum, then it means that the closed string degrees of freedom can be represented (non–perturbatively) in terms of the open string ones. This is an exciting possibility that has not been methodically explored so far.

2. Open string field theory

Before we come to the formal definition of string field theory, i.e. second quantized string theory, we need a short summary of first quantized open string theory.

2.1. First quantized open strings

First quantized open string theory in the critical dimension D=26 is formulated in terms of quantum oscillators α_n^μ, $-\infty < n < \infty$, $\mu = 0, 1, \ldots, 25$, which come from the mode expansion of the string scalar field

$$X^\mu(z) = \frac{1}{2}x^\mu - \frac{i}{2}p^\mu \ln z + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}z^{-n}$$

having set the characteristic square length of the string α' to 1. They satisfy the algebra $[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{n+m,0}$, η being the space–time Minkowski metric. The vacuum is defined by $\alpha_n^\mu|0\rangle = 0$ for $n > 0$ and $p^\mu|0\rangle = 0$. The states of the theory are constructed by applying to the vacuum the remaining quantum oscillators $\alpha_n^{\mu\dagger} = \alpha_{-n}^\mu$, with $n > 0$. Any such state $|\phi\rangle$ is given a momentum k^μ by multiplying it by the eigenstate e^{ikx}, and will be denoted by $|\phi, k\rangle$. In order for such states to be physical they must satisfy the conditions

$$L_n^{(X)}|\phi, k\rangle = 0, \quad n > 0, \qquad (L_0^{(X)} - 1)|\phi, k\rangle = 0 \tag{1}$$

where $L_n^{(X)}$ are the matter Virasoro generators

$$L_n^{(X)} = \frac{1}{2} : \sum_{k=-\infty}^{\infty} \alpha_{n-k}^\mu \alpha_k^\nu : \eta_{\mu\nu} \tag{2}$$

where we have set $\alpha_0 = p$ and :: denotes normal ordering. Conditions (1) are the quantum translation of the classical vanishing of the energy–momentum tensor.

The conditions (1) define the physical spectrum of the theory (in D=26). All the states are ordered according to the level, the level being a natural number specified by the eigenvalue of $L_0^{(X)} + L_0^{(gh)} - p^2$. The lowest lying state (level 0) is the tachyon represented by the vacuum with momentum k and square mass $M^2 = -1$. The next (level 1) is the massless vector state $\zeta_\mu \, \alpha_{-1}^\mu |0\rangle e^{ikx}$ with $k^2 = 0$ and $\zeta \cdot k = 0$, which is interpreted as a gauge field. The other states are all massive, with increasing masses proportional to the Planck mass square.

To each of these states is associated a vertex operator. For instance, to the tachyon we associate $V_t(k) =: e^{ik\cdot X}$:; to the vector state $V_A(k,\zeta) =: \zeta \cdot \dot{X} e^{ik\cdot X}$:, where the dot on top of X denotes the tangent derivative with respect to the world–sheet boundary (the real axis in the z UHP); and so on. In this way one can formulate rules to calculate any kind of amplitude of these operators $\langle V_1(k_1) \dots V_N(k_N)\rangle$, as far as these amplitudes are *on shell*. At low energy ($\alpha' \to 0$) such amplitudes, as expected, reproduce those of the corresponding field theory (for instance, the amplitudes of V_A reproduce the amplitudes of a Maxwell field theory). If we want to compute *off–shell* amplitudes, in general we have to resort to a field theory of strings. This was one of the original motivations for introducing string field theory.

So far we have ignored ghosts. In fact the b, c ghosts, which come from the gauge fixing of reparametrization invariance via the Faddeev–Popov recipe, play a minor role in perturbative string theory. They play a much more important role in SFT. They are also expanded in modes c_n and b_n and one can construct the corresponding Virasoro generators

$$L_n^{(gh)} =: \sum_k (2n+k)\, b_{-k} c_{k+n} : \tag{3}$$

Both (2) and (3) obey the same Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n) \tag{4}$$

The central charge c equals the number of X fields in the matter case (i.e. 26), while it equals -26 in the case of the b, c ghosts. So the total central charge vanishes in D=26. This guarantees the absence of any trace anomaly, and therefore consistency of the bosonic string theory as a gauge theory.

The previous results about ghosts and critical dimension, can be usefully reformulated in terms of BRST symmetry and its charge Q. Q is defined by

$$Q = \sum_n : c_n \left(L_n^{(X)} + \frac{1}{2} L_n^{(gh)} \right) : \tag{5}$$

It is hermitean $Q^\dagger = Q$ and its basic property is nilpotency

$$Q^2 = 0$$

in critical dimension. The study of the physical spectrum can be reformulated in terms of the cohomology of Q: *the physical states of perturbative string theory are the states of ghost number 1 that are annihilated by* Q, defined up to states obtained by acting with Q on any state of ghost number 0. They can be represented by the old physical states $|\phi, k\rangle$ tensored with the ghost factor $c_1|0\rangle$.

With this at hand we can now turn to string field theory.

2.2. The SFT action and star product

The open string field theory action proposed by E.Witten years ago, [1], is

$$\mathcal{S}(\Psi) = -\frac{1}{g_o^2}\int\left(\frac{1}{2}\Psi * Q\Psi + \frac{1}{3}\Psi * \Psi * \Psi\right). \tag{6}$$

with equation of motion

$$Q\Psi + \Psi * \Psi = 0 \tag{7}$$

The above action is clearly reminiscent of the Chern–Simons action in 3D. In this expression Ψ is the string field. It can be understood either as a classical functional of the open string configurations $\Psi(x^\mu(z))$, or as a vector in the Fock space of states of the open string theory. Although the first representation is more pictorial, the second is far more effective from a practical viewpoint. In the following we will consider for simplicity only this second point of view. In the field theory limit it makes sense to represent Ψ as a superposition of Fock space states with ghost number 1, with coefficient represented by (infinite many) local fields,

$$|\Psi\rangle = (\phi(x) + A_\mu(x)a_1^{\mu\dagger} + \ldots)c_1|0\rangle. \tag{8}$$

The BRST charge Q is the same as the one introduced above for the first quantized string theory.

One of the most fundamental ingredients is the star product. Physically it represents the string interaction, that is the process of two strings coming together to form a third string. More precisely the product of two string fields Ψ_1, Ψ_2 represents the process of identifying the right half of the first string with the left half of the second string and integrating over the overlapping degrees of freedom, to produce a third string which corresponds to $\Psi_1 * \Psi_2$. This can be implemented in different ways, either using the classical string functional (as in the original formulation by Witten), or using the three string vertex, or the conformal field theory language, [15].

Finally the integration in (6) corresponds to bending the left half of the string over the right half and integrating over the corresponding degrees of freedom in such a way as to produce a number.

The following rules are to be complied with

$$\begin{aligned}
&Q^2 = 0, \\
&\int Q\Psi = 0, \\
&(\Psi_1 * \Psi_2) * \Psi_3 = \Psi_1 * (\Psi_2 * \Psi_3), \\
&Q(\Psi_1 * \Psi_2) = (Q\Psi_1) * \Psi_2 + (-1)^{|\Psi_1|}\Psi_1 * (Q\Psi_2),
\end{aligned} \tag{9}$$

where $|\Psi|$ is the Grassmannality of the string field Ψ, which, for bosonic strings, coincides with the ghost number. The action (6) is invariant under the BRST transformation

$$\delta\Psi = Q\Lambda + \Psi * \Lambda - \Lambda * \Psi. \tag{10}$$

Finally, the ghost numbers of the various objects $Q, \Psi, \Lambda, *, \int$ are $1, 1, 0, 0, -3$, respectively.

Let us now see in more detail how to implement the star product. Let us consider three unit semi-disks in the upper half z_a $(a = 1, 2, 3)$ plane. Each one represents the string freely propagating in semicircles from the origin (world-sheet time $\tau = -\infty$) to the unit circle $|z_a| = 1$ $(\tau = 0)$, where the interaction is supposed to take place. We map each unit semi-disk to a 120° wedge of the unit disk via the following conformal maps:

$$f_a(z_a) = \alpha^{2-a} f(z_a)\,,\; a = 1, 2, 3, \tag{11}$$

where

$$f(z) = \Big(\frac{1 + iz}{1 - iz}\Big)^{\frac{2}{3}}. \tag{12}$$

Here $\alpha = e^{\frac{2\pi i}{3}}$. In this way the three semi-disks are mapped to non-overlapping (except along the edges) regions in such a way as to fill up a unit disk centered at the origin. The curvature is zero everywhere except at the center of the disk, which represents the common midpoint of the three strings in interaction, see Fig.1

The interaction vertex is defined by means of a correlation function on the unit disk D in the following way

$$\int \psi * \phi * \chi = \langle f_1 \circ \psi(0)\, f_2 \circ \phi(0)\, f_3 \circ \chi(0)\rangle_D \tag{13}$$

So, calculating the star product amounts to evaluating a three point function on the unit disk.

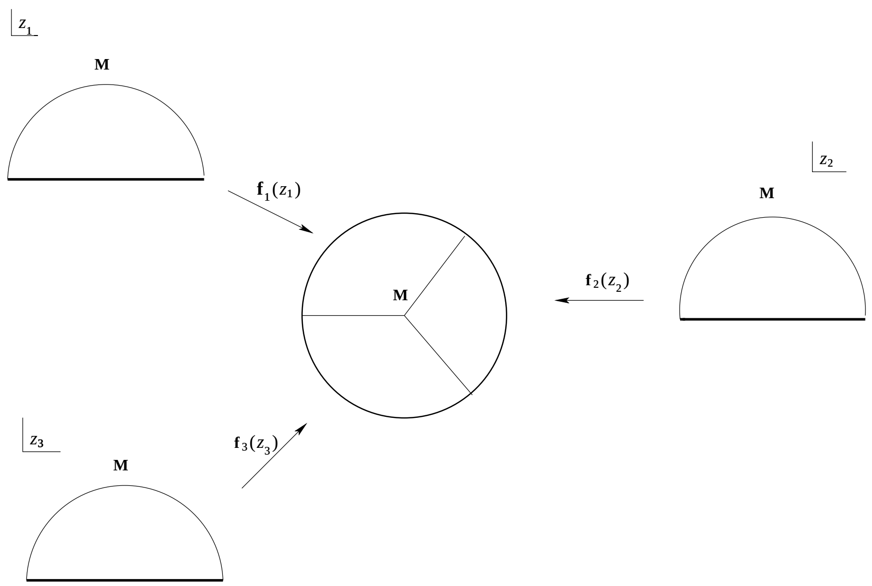

Figure 1: *The conformal maps from the three unit semi-disks to the three-wedges unit disk*

3. Tachyon condensation

Following the rules of the previous section it is possible to explicitly compute the action (6). For instance, in the low energy limit, where the string field may be assumed to take the form (8), the action becomes an integrated function F of an infinite series of local polynomials (kinetic and potential terms) of the fields involved in (8):

$$\mathcal{S}(\Psi) = \int d^{26}x\, F(\varphi_i, \partial\varphi_i, ...). \tag{14}$$

To limit the number of terms one has to limit the gigantic BRST symmetry of OSFT, by choosing a gauge, which is usually the Feynman–Siegel gauge: this means that we limit ourselves to the states that satisfy the condition: $b_0|\Psi\rangle = 0$

Still the action with all the infinite sets of fields contained in Ψ remains unwieldy. As it turns out, it makes sense to limit the number of fields in Ψ, provided we insert all the fields up to a certain level. This is called *level truncation* and turns out to be an excellent approximation and regularization scheme in SFT. Let us see this in more detail for a string field which includes the tachyon $\phi(x)$ and the vector field $A_\mu(x)$. The action turns out to be, [11],

$$\mathcal{S}_{(0,1)} = \frac{1}{g_o^2}\int d^{26}x \Big(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\phi^2 - \frac{1}{3}\beta^3\phi^3 - \frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu \\ - \beta\tilde{\phi}\tilde{A}_\mu\tilde{A}^\mu - \frac{\beta}{2}(\partial_\mu\partial_\nu\tilde{\phi}\tilde{A}^\mu\tilde{A}^\nu + \tilde{\phi}\partial_\mu\tilde{A}^\nu\partial_\nu\tilde{A}^\mu - 2\partial_\mu\tilde{\phi}\partial_\nu\tilde{A}^\mu\tilde{A}^\nu)\Big) \tag{15}$$

where $\beta = \frac{3\sqrt{3}}{4}$ is a recurrent number in SFT. One can see the kinetic term for the tachyon and the gauge field (the latter is in the gauge fixed form because the Feynman–Siegel gauge corresponds in the field theory language to the Lorentz gauge) and the 'wrong' mass term for the tachyon. The fields appearing in the interactions terms carry a tilde. This means, for any field φ

$$\tilde{\varphi}(x) = e^{-\ln(\beta^{-1}\partial_\mu\partial^\mu)}\varphi(x)$$

Incidentally, the fact that the interaction is formulated in terms of tilded fields is a manifestation of the strong (exponential) convergence properties of string theory in the UV.

Let us now consider the potential and study its minimum. We remind the reader that this theory is supposed to represent the open strings attached to a space–filling D–brane, the D25–brane. It may also represent lower dimensional branes. In the CFT language such configurations are described by boundary CFT's. The first important remark (Sen 1998) is that this potential is universal, it does not depend on the details of the theory, i.e. on a particular boundary conformal field theory.

Let us concentrate on the D25–brane and evaluate the total energy of the system brane + string modes. The brane has its intrinsic energy, whose density is the tension τ, which in our conventional units ($\alpha' = 1$), is given by $\tau = \frac{1}{2\pi^2 g_o^2}$. The string modes are represented by the action and, in a static situation, their total energy is given by – the action. We precisely wish to study this system in the vacuum. Since we want Lorentz invariance, only Lorentz scalars can acquire a VEV. Therefore in (15) one must set the tensor fields and all the derivatives to 0. Setting $\langle\phi\rangle = t$, what remains of the action (divided by the total volume) can be written in terms of the function $u(t)$ as follows

$$-\frac{S}{V} \equiv \tau\, u(t) = 2\pi^2\left(-\frac{1}{2}t^2 + \frac{1}{3}\beta^3 t^3\right) \tag{16}$$

This is the total tachyon potential energy density extracted from the action.

The total energy of the system will be given by the sum of (16) and the D25–brane tension, i.e.

$$U(t) = \tau(1 + u(t)) \tag{17}$$

This potential is cubic, and it is easy to determine both local maximum and minimum. The latter is given by

$$t = t_0 = \frac{1}{\beta^3}, \qquad u(t) \approx -0.684 \tag{18}$$

Let us recall that the first conjecture by Sen is that the tachyonic energy should exactly compensate for the D25–brane tension. Therefore (18) does not match this result, but we should remember that ours has been a very rough approximation, since we have retained only two fields, the tachyon and the Maxwell field. It can be shown that by adding more and more fields to the string fields Ψ, that is truncating it at a higher level, the value of $u(t_0)$ gets closer and closer to -1. The asymptotic situation is represented in Fig.2

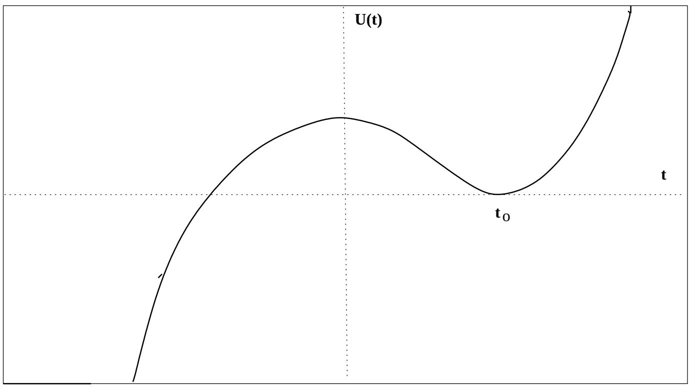

Figure 2: *The tachyon potential*

This was historically the first evidence that the first Sen's conjecture is correct.

4. The analytic solution

In this section I will explain how the first analytic solution to the SFT equation of motion (7) was found, [7] (see also [9]) This solution is a string state that specifies the (locally) stable vacuum, to be identified as the closed string vacuum. In the oversimplified language of Fig. 2 it would correspond to $|\Psi_0\rangle = t_0 c_1|0\rangle$, but it actually identifies the vev of all the infinite many scalar fields that feature in the most general string field.

To start with I have to introduce one of the important ingredients of this solution, the wedge states.

4.1. Wedge states and the new coordinate patch

Wedge states are particular surface states. The latter are states simply defined by a map from the half–disk to the unit disk or, equivalently, to the upper half plane. The definition is as follows: take any map f from the half–disk to a surface Σ (inscribed in the unit disk or in the UHP); consider any field ϕ and the state $|\phi\rangle = \phi(0)|0\rangle$ in the Fock space of the theory; then the surface state $\langle S|$ is defined by

$$\langle S|\phi\rangle = \langle f \circ \phi\rangle_\Sigma \tag{19}$$

The definition is implicit and may seem at first not very handy, but one can often reduce the calculation to very simple test states $|\phi\rangle$.

Wedge states are particularly simple. Their defining functions are

$$f_r(z) = \left(\frac{1+iz}{1-iz}\right)^{\frac{2}{r}} \tag{20}$$

where, for simplicity, we take r to be a positive integer. This means that the image of the map is a wedge of angle $\frac{2\pi}{r}$ in the unit disk. They can be shown to satisfy the recursion relation

$$|r\rangle \star |s\rangle = |r+s-1\rangle \tag{21}$$

In particular we see that calling $|\Xi\rangle$ the result of taking $r \to \infty$ in $|r\rangle$, we recover $\Xi^2 = \Xi$. This may seem formal, but it can be shown to give rise precisely to the sliver, which is a surface state defined by a wedge of vanishing angle. So, in particular, wedge states approximate the sliver.

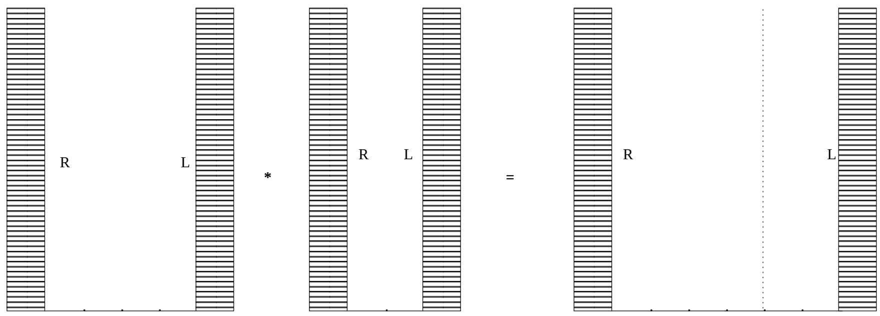

Figure 3: *Star product of two wedge states* $|3\rangle \star |2\rangle = |4\rangle$

The star product of wedge states takes a particularly simple form if we use the coordinate $\tilde{z} = \arctan z$ (this we be referred to in the sequel as the arctan frame). In this new representation a wedge state $|r\rangle$ is a cylinder in the $\tilde{z}$ UHP, see fig.3. It is in fact an infinite strip in the imaginary direction of width $r\frac{\pi}{2}$. It is formed by two external strips of width $\frac{\pi}{4}$ each (the ruled strips in the figure), and an internal strip of width $(r-1)\frac{\pi}{2}$. The rightmost and leftmost sides are identified so as to form a cylinder. The star product of two such states is simply obtained by dropping the rightmost ruled strip of the first state and the leftmost ruled strip of the second and gluing the two cut cylinders along the dashed line in Fig.3. In this language the wedge state with $r = 2$ corresponds to the vacuum $|0\rangle$.

Pure wedge states, as we have just described them, are not enough to describe the analytic solution we are looking for. We need wedge states with insertion, that is wedge states with the insertion of an operator at some point of the unruled patches. The $|n\rangle$ wedge state itself can be seen as such.

$$|n\rangle = \left(\frac{2}{n}\right)^{\mathcal{L}_0^\dagger} |0\rangle \tag{22}$$

where $\mathcal{L}_0$ will be introduced in a moment.

These states will play a major role in what follows. What we need now is exploit the new coordinate $\tilde{z} = \arctan z$ to get a few basic definitions and relations. To start with we define the Virasoro generators in the new coordinate patch

$$\mathcal{L}_0 = \oint \frac{d\tilde{z}}{2\pi i} \tilde{z}\, T_{\tilde{z}\tilde{z}}(\tilde{z})$$

that is

$$\mathcal{L}_0 = L_0 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2-1} L_{2k} \tag{23}$$

as well as $\mathcal{L}_{\pm 1}$. They satisfy $[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m}$.

Other useful operators are

$$\begin{aligned}
\mathcal{B}_0 &= b_0 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2-1} b_{2k} \\
B_1 &= b_1 + b_{-1}
\end{aligned}$$

and

$$\begin{aligned}
B \equiv B_1^L &= \frac{1}{2} B_1 + \frac{1}{\pi}\left(\mathcal{B}_0 + \mathcal{B}_0^\dagger\right) \\
B_1^R &= \frac{1}{2} B_1 - \frac{1}{\pi}\left(\mathcal{B}_0 + \mathcal{B}_0^\dagger\right)
\end{aligned}$$

Using $K_1 = L_1 + L_{-1}$ we can introduce

$$K \equiv K_1^L = \frac{1}{2}K_1 + \frac{1}{\pi}\left(\mathcal{L}_0 + \mathcal{L}_0^\dagger\right)$$
$$K_1^R = \frac{1}{2}K_1 - \frac{1}{\pi}\left(\mathcal{L}_0 + \mathcal{L}_0^\dagger\right)$$

For instance we have the 'semi–derivation' rules

$$K_1^L(\Psi_1 \star \Psi_2) = (K_1^L\Psi_1) \star \Psi_2$$
$$K_1^R(\Psi_1 \star \Psi_2) = \Psi_1 \star (K_1^R\Psi_2)$$

and the wedge states can also be written as

$$|n\rangle = e^{\frac{\pi}{2}(n-1)K}|1\rangle$$

From this equation and (22) we see that it makes sense to consider n a real variable rather than an integer, and therefore also to differentiate with respect to it.

4.2. The solution

Schnabl chose the gauge $\mathcal{B}_0|\Psi\rangle = 0$, rather than the Feynman–Siegel one. He than made the ansatz

$$\Psi = \lim_{N\to\infty}\left(\sum_{n=0}^{N}\psi'_n - \psi_N\right) \tag{24}$$

where

$$\psi_n = c_1|0\rangle \star B|n\rangle \star c_1|0\rangle \tag{25}$$

and the prime denotes derivative with respect to n. The state ψ_n is made out of wedge states with insertions of the field c and of B. In particular for $n = 0$ we have

$$\psi_0 = (cBc)(0)|0\rangle, \qquad \psi'_0 = (cBKc)(0)|0\rangle$$

We remark that in the RHS of (24) the second term $-\psi_N$ is added only for regularization purposes.

The solution is obtained as a limit and it is constructed as

$$\Psi_\lambda = \sum_{n=0}^{\infty}\lambda^{n+1}\psi'_n, \tag{26}$$

This is a pure gauge solution (action=0) for $\lambda < 1$, but it is not pure gauge anymore for $\lambda = 1$ and it is the good solution. We will not prove it here. Rather we concentrate on the evidence about first Sen's conjecture.

4.3. First and second Sen's conjectures

From the equation of motion we get

$$\langle \Psi, Q\,\Psi\rangle = -\langle \Psi, \Psi \star \Psi\rangle \tag{27}$$

This equation has to be explicitly checked over the solution (24) – a rather nontrivial task –, because one of the subtleties of SFT is that, even if $|\Psi\rangle$ is a solution to the equation of motion, it is not automatically guaranteed that (27) holds.

On the other hand, from the explicit form of the solution one gets

$$\langle \Psi, Q\,\Psi\rangle = -\frac{3}{\pi^2}$$

Therefore, finally, the total energy of the string modes is (V is the total 26–th dimensional volume):

$$E = -\frac{S}{V} = \frac{1}{g_o^2 V}\left(\frac{1}{2}\langle \Psi, Q\,\Psi\rangle + \frac{1}{3}\langle \Psi, \Psi \star \Psi\rangle\right) = -\frac{1}{2\pi^2 g_0^2} \tag{28}$$

which is precisely the negative of the D25–brane tension τ.

Let us now pass to briefly illustrate the proof of the second conjecture, [8]. The purpose is to show that the cohomology around Schnabl's solution is trivial. Relabeling Schnabl's solution as Ψ_0, we are looking now for solutions to (7) of the type $\Psi_0 + \psi$, linearized on ψ. It is easy to see that the relevant (linearized) equation of motion is

$$\mathcal{Q}\psi \equiv Q\psi + \Psi_0 \star \psi - (-1)^{|\psi|}\psi \star \Psi_0 \tag{29}$$

This defines a new BRST operator $\mathcal{Q}$ (indeed $\mathcal{Q}^2 = 0$) and defines the cohomology around Schnabl's solution. The purpose is to prove that this cohomology is empty.

Let us introduce the symbol

$$W_r = |r+1\rangle$$

Next let us define the state

$$A = -\frac{2}{\pi}\, B \int_0^1 W_r\, dr \tag{30}$$

We make use of the fact that wedge states can be defined for any real label r, not just for an integral r. It is possible to prove that

$$\mathcal{Q}A = |1\rangle \tag{31}$$

where the RHS represents the wedge state with $r = 1$. This is the identity state and satisfies

$$|1\rangle \star \Phi = \Phi \star |1\rangle = \Phi$$

for any string field Φ.

Now suppose ψ satisfies $\mathcal{Q}\psi = 0$, then, using these results, we get

$$\mathcal{Q}(A \star \psi) = (\mathcal{Q}A) \star \psi - A \star (\mathcal{Q}\psi) = |1\rangle \star \psi = \psi$$

which means that ψ is BRST trivial. This is a very general result. It implies not only that the cohomology of ghost number 1 is trivial (i.e., there is no physical perturbative string mode in the new vacuum), but that the cohomology is trivial for any ghost number state.

4.4. The Erler–Schnabl solution

An interesting variant of the Schnabl solution has been proposed recently by T.Erler and M.Schnabl, [17]. To better describe it we shift from the language of string fields and operators K, B, c used so far to an 'algebra with operator' language defined as follows. Let us set

$$K\frac{\pi}{2}K_1^L|I\rangle, \qquad B = \frac{\pi}{2}B_1^L|I\rangle, \qquad c = c\left(\frac{1}{2}\right)|I\rangle, \tag{32}$$

in the so–called sliver frame (obtained by mapping the UHP to an infinite cylinder C_2 of circumference 2, by the sliver map $f(z) = \frac{2}{\pi}\arctan z$). Then, with respect to the star product (understood from now on), these states form the algebra

$$\{B, c\} = 1, \qquad KB = BK, \qquad [K, c] = \partial c, \qquad \{B, \partial c\} = 0, \tag{33}$$

where Q operates as follows

$$QB = K, \qquad Qc = c\partial c \tag{34}$$

In terms of this algebra with operator, the new solution, [17], is given by

$$\psi_0 = \frac{1}{1+K}c(1+K)Bc = c - \frac{1}{1+K}Bc\partial c, \tag{35}$$

and can be formally obtained via a 'gauge transformation' of the perturbative vacuum

$$\psi_0 = U_0 Q U_0^{-1} \tag{36}$$

$$U_0 = 1 - \frac{1}{1+K}Bc \tag{37}$$

$$U_0^{-1} = 1 + \frac{1}{K}Bc. \tag{38}$$

This gauge transformation is in fact singular (see above) and this is the reason why the solution is nontrivial. The energy of this solution turns out to be the correct one (28). Not only the solution but also the homotopy operator A is very simple: $A = \frac{B}{K+1}$.

As we will see below, the Erler–Schnabl solution lends itself to a rather simple matter deformation, which turns out to be the searched for lump solution.

5. The third conjecture

The third conjecture predicts the existence of lower dimensional solitonic solutions or lumps, interpreted as Dp–branes, with $p < 25$. These solutions bring along the breaking of translational symmetry and background independence. The evidence for the existence of such solutions collected in the past years is overwhelming. It has been possible to find them with approximate methods or with exact methods in related theories. Probably the most suggestive and significant solutions were found in the framework of the so called vacuum string field theory (VSFT). VSFT, [18, 19], is an approximate version of Witten's open SFT which is thought to describe the latter at the minimum of the tachyonic potential. In this theory matter and ghost sectors split and essentially only the matter part is relevant. The lower dimensional branes make their appearance as projector projectors of the star algebra. This results in a very elegant formulation, [16, 13]. Although the relation of VSFT to the true theory is far from clear, these results have been very inspiring in the search for analytic solutions.

5.1. Analytic lump solutions

In a recent paper, [20], a general method has been proposed to obtain new exact analytic solutions in open string field theory, and in particular solutions that describe inhomogeneous tachyon condensation. The method consists in translating an exact renormalization group (RG) flow generated in a two–dimensional world–sheet theory by a relevant operator, to the language of OSFT. The so-constructed solution is a deformation of the Erler-Schnabl solution described above. It has been shown in [20] that, if the operator has suitable properties, the solution will describe tachyon condensation in specific space directions, thus representing the condensation of a lower dimensional brane. In the following, after describing the general method, we will focus on a particular solution, generated by an exact RG flow analyzed first by Witten, [21]. In [20] it has been shown that this solution satisfies the closed string overlap condition and, on the basis of the analysis carried out in the framework of 2D CFT in [22], we expect it to describe a D24 brane, with the correct ratio of tension with the starting D25 brane.

Let us see how to construct such kind of lump solutions. To start with we enlarge the K, B, c algebra by adding a (relevant) matter operator

$$\phi = \phi\left(\frac{1}{2}\right)|I\rangle. \tag{39}$$

with the properties

$$[c,\phi]=0,\qquad [B,\phi]=0,\qquad [K,\phi]=\partial\phi, \tag{40}$$

such that Q has the following action:

$$Q\phi = c\partial\phi + \partial c\delta\phi. \tag{41}$$

It can be easily proven that

$$\psi_\phi = c\phi - \frac{1}{K+\phi}(\phi-\delta\phi)Bc\partial c \tag{42}$$

does indeed satisfy the OSFT equation of motion

$$Q\psi_\phi + \psi_\phi\psi_\phi = 0 \tag{43}$$

It is clear that (42) is a deformation of the Erler–Schnabl solution, which can be recovered for $\phi = 1$.

Much like in the Erler-Schnabl case, we can view this solution as a singular gauge transformation

$$\psi_\phi = U_\phi Q U_\phi^{-1} \tag{44}$$

where

$$U_\phi = 1 - \frac{1}{K+\phi}\phi Bc,\qquad U_\phi^{-1} = 1 + \frac{1}{K}\phi Bc, \tag{45}$$

In order to prove that (42) is a solution, one demands that $(c\phi)^2 = 0$, which requires the OPE of ϕ at nearby points to be not too singular.

It is instructive to write down the kinetic operator around this solution. With some manipulation, using the K, B, c, ϕ algebra it is possible to show that

$$\mathcal{Q}_{\psi_\phi}\frac{B}{K+\phi} = Q\frac{B}{K+\phi} + \left\{\psi_\phi, \frac{B}{K+\phi}\right\} = 1.$$

So, unless the homotopy–field $\frac{B}{K+\phi}$ is singular (as is the case for $\frac{B}{K}$ and the original Q), the solution has trivial cohomology, which is the defining property of the tachyon vacuum [8]. On the other hand, in order for the solution to be well defined, the quantity $\frac{1}{K+\phi}(\phi-\delta\phi)$ should be well defined. In full generality we thus have a new nontrivial solution if

- $\frac{1}{K+\phi}$ is divergent
- $\frac{1}{K+\phi}(\phi-\delta\phi)$ is finite.

We will now give some heuristic sufficient conditions for ϕ in order to comply with the above requirement. To be concrete we parametrize the worldsheet RG flow, referred to above, by a parameter u, where $u = 0$ represents the UV and $u = \infty$ the IR, and label ϕ by ϕ_u, with $\phi_{u=0} = 0$. Then we require for ϕ_u the following properties under the coordinate rescaling $f_t(z) = \frac{z}{t}$

$$f_t \circ \phi_u(z) = \frac{1}{t}\,\phi_{tu}\left(\frac{z}{t}\right). \tag{46}$$

and, most important, that the partition function

$$g(u) \equiv Tr[e^{-(K+\phi_u)}] = \left\langle e^{-\int_0^1 ds\,\phi_u(s)}\right\rangle_{C_1}, \tag{47}$$

satisfies the asymptotic finiteness condition

$$\lim_{u\to\infty}\left\langle e^{-\int_0^1 ds\,\phi_u(s)}\right\rangle_{C_1} = \texttt{finite}. \tag{48}$$

Barring subtleties, this should satisfy the two above conditions, i.e. guarantee not only the regularity of the solution but also its 'non-triviality', in the sense that if this condition is satisfied, it cannot fall in the same class as the ES tachyon vacuum. solution.

We will consider in the sequel a specific relevant operator ϕ_u and the corresponding SFT solution. It is based on the exact RG flow studied by Witten in [21], see also [22], and based on the operator (defined in the cylinder C_T of width T in the arctan frame)

$$\phi_u(s) = \frac{u}{2}(X^2(s) + 2\,\mathrm{ln}u + 2A) \tag{49}$$

where A is a constant first introduced in [8]. In C_1 we have

$$\phi_u(s) = \frac{u}{2}(X^2(s) + 2\,\mathrm{ln}Tu + 2A) \tag{50}$$

and on the unit disk D,

$$\phi_u(\theta) = \frac{u}{2}(X^2(\theta) + 2\ln\frac{Tu}{2\pi} + 2A) \tag{51}$$

If we set

$$g_A(u) = \langle e^{-\int_0^1 ds\,\phi_u(s)}\rangle_{C_1} \tag{52}$$

we have

$$g_A(u) = \langle e^{-\frac{1}{2\pi}\int_0^{4\pi} d\theta\,u\left(X^2(\theta)+2\ln\frac{u}{2\pi}+2A\right)}\rangle_D$$

According to [21],

$$g_A(u) = Z(u)e^{-u(\ln\frac{u}{2\pi}+A)} \tag{53}$$

Requiring finiteness for $u \to \infty$ we get $A = \gamma - 1 + \ln 2\pi$, which implies

$$g_A(u) \equiv g(u) = \frac{1}{2\sqrt{\pi}}\sqrt{u}\Gamma(u)e^{u(1-\ln u)} \tag{54}$$

and

$$\lim_{u\to\infty} g(u) = \frac{1}{\sqrt{2}} \tag{55}$$

Moreover, as it turns out, $\delta\phi_u = -u$, and so:

$$\phi_u - \delta\phi_u = u\partial_u\phi_u(s) \tag{56}$$

Therefore the ϕ_u just introduced satisfies all the requested properties. According to [22], the corresponding RG flow in BCFT reproduces the correct ratio of tension between D25 and D24 branes. Consequently $\psi_u \equiv \psi_{\phi_u}$ should represent a D24 brane solution.

In SFT there are two gauge invariant quantities we can extract from a solution like ψ_u. One is the closed string overlap, the other is the energy. The simplest gauge invariant quantity that can be explicitly computed is the closed string overlap, i.e. the overlap between the solution and an on–shell closed string state inserted at the midpoint of the identity string field. According to Ellwood, [23], this quantity should be equal to the shift in the closed string one–point function between the new BCFT represented by the new solution and the reference BCFT represented by the perturbative vacuum. In full generality, if ψ_1 represents the new $BCFT_1$, expressing everything on a canonical cylinder of width 1, we expect to find

$$Tr[V_c\,\psi_1] = \langle V_c(i\infty)c(0)\rangle_{C_1}^{(BCFT_0)} - \langle V_c(i\infty)c(0)\rangle_{C_1}^{(BCFT_1)}\,. \tag{57}$$

where V_c is the vertex operator generating the on-shell closed string state. This relation has in fact been proved to hold for our solution ψ_u in [20].

As for the energy its expression for the lump solution was determined in [20] by evaluating a three–point function on the cylinder C_T of circumference T in the arctan frame. It equals $-\frac{1}{6}$ times the following expression

$$\begin{aligned}\langle\psi_u\psi_u\psi_u\rangle = &-\int_0^\infty dt_1dt_2dt_3\mathcal{E}_0(t_1,t_2,t_3)u^3g(uT)\Bigg\{\Big(-\frac{\partial_{uT}g(uT)}{g(uT)}\Big)^3\\ &+\frac{1}{2}\Big(-\frac{\partial_{uT}g(uT)}{g(uT)}\Big)\Big(G^2_{uT}(\frac{2\pi t_1}{T})+G^2_{uT}(\frac{2\pi(t_1+t_2)}{T})+G^2_{uT}(\frac{2\pi t_2}{T})\Big)\\ &+G_{uT}(\frac{2\pi t_1}{T})G_{uT}(\frac{2\pi(t_1+t_2)}{T})G_{uT}(\frac{2\pi t_2}{T})\Bigg\}\end{aligned} \tag{58}$$

Here $g(u)$ is as above, and represents the partition function of the underlying matter CFT on the boundary of the unit disk. $G_u(\theta)$ represents the correlator on the boundary, first determined by Witten, [21]:

$$G_u(\theta) = \frac{1}{u} + 2\sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k+u} \tag{59}$$

Finally $\mathcal{E}_0(t_1, t_2, t_3)$ represents the ghost three–point function in C_T.

$$\begin{aligned} \mathcal{E}_0(t_1, t_2, t_3) &= \langle Bc\partial c(t_1+t_2)\partial c(t_1)\partial c(0)\rangle_{C_T} \\ &= -\frac{4}{\pi}\sin\frac{\pi t_1}{T}\sin\frac{\pi(t_1+t_2)}{T}\sin\frac{\pi t_2}{T} \end{aligned} \tag{60}$$

This expression is expected to take the form

$$-\frac{1}{6}\langle\psi_u\,\psi_u\,\psi_u\rangle = -E^{(UV)} + E^{(IR)}, \tag{61}$$

where $E^{(UV)}$ represents the total energy of the Erler–Schnabl solution, while $E^{(IR)}$ is the lump energy. Since the energy $E^{(UV)}$ is the D25 brane tension times the volume, which is infinite, we expect to find a divergent contribution when $u \to 0$. This is in fact what happens. In [24] the expression (58) has been evaluated. It has been analytically computed up to the point permitted by our present mathematical tools. The divergent behaviour in the UV has been found. After subtracting it, the remaining expression has been numerically evaluated. The result is 0.050004. This is compatible with the expected value of the D24 brane tension[1]

$$T_{D24} = \frac{1}{2\pi^2} \tag{62}$$

within the errors of the numerical approximation.

References

[1] E.Witten, *Noncommutative Geometry and String Field Theory*, Nucl.Phys. **B268** (1986) 253.

[2] A.Sen *Tachyon Condensation on the Brane Antibrane System* JHEP **9808** (1998) 012, [hep-th/9805170]. *BPS D–branes on Non–supersymmetric Cycles*, JHEP **9812** (1998) 021, [hep-th/9812031].

[3] A.Sen *Descent Relations among Bosonic D–Branes*, Int.J.Mod.Phys. **A14** (1999) 4061, [hep-th/9902105].

[4] A. Sen, *Rolling tachyon,* JHEP **0204** (2002) 048 [arXiv:hep-th/0203211].

[5] A. Sen, *Open-closed duality: Lessons from matrix model,* Mod. Phys. Lett. A **19** (2004) 841 [arXiv:hep-th/0308068].

[1] Eq.(62) takes in to account also a change of normalization with respect to [17], see [24].

[6] A. Sen, *Open and closed strings from unstable D-branes,* Phys. Rev. D **68** (2003) 106003, [arXiv:hep-th/0305011].

[7] M. Schnabl, *Analytic solution for tachyon condensation in open string field theory,* Adv. Theor. Math. Phys. **10** (2006) 433 [arXiv:hep-th/0511286].

[8] I. Ellwood and M. Schnabl, *Proof of vanishing cohomology at the tachyon vacuum,* JHEP **0702** (2007) 096 [arXiv:hep-th/0606142].

[9] Y. Okawa, *Comments on Schnabl's analytic solution for tachyon condensation in Witten's open string field theory,* JHEP **0604** (2006) 055 [arXiv:hep-th/0603159].

[10] W. Taylor and B. Zwiebach, *D-branes, tachyons, and string field theory,* [hep-th/0311017].

[11] K.Ohmori, *A review of tachyon condensation in open string field theories*, [hep-th/0102085].

[12] I.Ya.Aref'eva, D.M.Belov, A.A.Giryavets, A.S.Koshelev, P.B.Medvedev, *Non-commutative field theories and (super)string field theories*, [hep-th/0111208].

[13] L.Bonora, C.Maccaferri, D.Mamone and M.Salizzoni, *Topics in string field theory,* [hep-th/0304270].

[14] E. Fuchs and M. Kroyter, *Analytical Solutions of Open String Field Theory*, [hep-th/0807.4722].

[15] A.Leclair, M.E.Peskin, C.R.Preitschopf, *String Field Theory on the Conformal Plane. (I) Kinematical Principles*, Nucl.Phys. **B317** (1989) 411.

[16] L.Rastelli, A.Sen and B.Zwiebach, *Classical solutions in string field theory around the tachyon vacuum*, Adv. Theor. Math. Phys. **5** (2002) 393 [hep-th/0102112].

[17] T. Erler and M. Schnabl, *A Simple Analytic Solution for Tachyon Condensation,* JHEP **0910** (2009) 066 [arXiv:0906.0979 [hep-th]]

[18] L.Rastelli, A.Sen and B.Zwiebach, *String field theory around the tachyon vacuum*, Adv. Theor. Math. Phys. **5** (2002) 353 [hep-th/0012251].

[19] L.Rastelli, A.Sen and B.Zwiebach, *Vacuum string field theory,* [hep-th/0106010].

[20] L. Bonora, C. Maccaferri and D. D. Tolla, *Relevant Deformations in Open String Field Theory: a Simple Solution for Lumps,* [arXiv:1009.4158 [hep-th]].

[21] E. Witten, *Some computations in background independent off-shell string theory,* Phys. Rev. D **47**, 3405 (1993) [arXiv:hep-th/9210065].

[22] D. Kutasov, M. Marino and G. W. Moore, *Some exact results on tachyon condensation in string field theory,* JHEP **0010**, 045 (2000) [arXiv:hep-th/0009148].

[23] I. Ellwood, *The closed string tadpole in open string field theory,* JHEP **0808** (2008) 063 [arXiv:0804.1131 [hep-th]].

[24] L. Bonora, C. Maccaferri and D. D. Tolla, *Energy of the analytic lump solution in SFT*, in preparation.