Linear Fractional Time Series*

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Abstract

Linear fractional (Möbius) transformations:

$$f(x) = \frac{a + bx}{c + dx}, \quad ad - bc \neq 0,$$

are related to many parts of pure mathematics and to some its applications. Parameters a, b, c, d may be some complex or p-adic numbers, or adeles. It is of a particular interest to consider linear fractional recurrences (LFR)

$$x_{n+1} = \frac{a + bx_n}{c + dx_n}, \quad n = 0, 1, 2...$$

as dynamical systems with discrete set of states x_n. We consider various properties of the above LFR in the role of time series, which are sequences of chronologically ordered data. Here index n represents equidistant discrete time variable and x_n is a quantity. Practically, the main task in the time series is to predict the future values of the x_n on

* This work is partially supported by the *Serbian Ministry of Science and Technological Development*, project No. 144032D.
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Key words and phrases: Financial time series, modeling, forecasting, linear fractional recurrences.
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the basis of the previous ones. To use LFR for an analysis of the real time series, e.g. financial time series, we explore possible changes in the space of parameters a, b, c, d. Comparison of our predicted values with concrete financial time series is encouraging.

1. Introduction

1.1. Time series

A time series is a sequence of data with a certain chronological ordering. It can be also understood as a realization of a random process. Namely, let $\{X_t, t \in \mathbb{R}\}$ be a given random process whose realization in certain moments is:

$$\begin{array}{ll} t = 1, 2, \dots, n & X_1, X_2, \dots X_n \\ t = 1, 2, \dots & X_1, X_2, \dots \\ t = 0, \pm 1, \pm 2, \dots & \dots X_{-2}, X_{-1}, X_0, X_1, X_2, \dots . \end{array}$$

Each of these strings is a time series. So in this sense, a time series is a string (finite or infinite) of random variables.

One of the most important tasks when studying time series is to predict their future values, both in the near and distant future. Due to a widespread and always topical need and call for the answer to such a question different methods have been developed which, depending on the nature of a time series, are all trying to produce the best possible answer to the following question: what is the value of a time series in a certain moment in the future (see, e.g. [1] and references therein)?

1.2. Linear fractional transformations

A linear fractional transformation (LFT) is defined as a function of the form

$$f(z) = \frac{\alpha_0 + \alpha_1 \cdot z}{\beta_0 + \beta_1 \cdot z},$$

where z, $\alpha_0, \alpha_1, \beta_0, \beta_1$ are usually complex numbers satisfying condition $\alpha_0 \cdot \beta_1 - \alpha_1 \cdot \beta_0 \neq 0$. This kind of mapping is also called homographic transformation or Möbius transformation. LFT defines one-to-one mapping of the extended complex plane $(\mathbb{C} \bigcup \{\infty\})$ onto itself. The set of all LFT transformations forms a group under composition called the Möbius group. Various linear fractional transformations and the corresponding groups play significant role in pure and applied mathematics.

1.3. Linear fractional recurrences

Let us define a linear fractional recurrence relation of the first order (LFR1) as

$$z_k = \frac{\alpha_0 + \alpha_1 \cdot z_{k-1}}{\beta_0 + \beta_1 \cdot z_{k-1}},$$

where $z_k, z_{k-1}, \alpha_0, \alpha_1, \beta_0, \beta_1$ are usually complex numbers satisfying $\alpha_0 \cdot \beta_1 - \alpha_1 \cdot \beta_0 \neq 0$.

Let a linear fractional recurrence relation of q-th order (LFRq) [2] be

$$z_k = \frac{\alpha_0 + \alpha_1 \cdot z_{k-q} + \alpha_2 \cdot z_{k-q+1} + \cdots + \alpha_q \cdot z_{k-1}}{\beta_0 + \beta_1 \cdot z_{k-q} + \beta_2 \cdot z_{k-q+1} + \cdots + \beta_q \cdot z_{k-1}}$$

$$= \frac{\alpha_0 + \sum\limits_{i=1}^{q} \alpha_i \cdot z_{k-q+i-1}}{\beta_0 + \sum\limits_{i=1}^{q} \beta_i \cdot z_{k-q+i-1}},$$

where $z_{k-i}, \alpha_i, \beta_i$, $(i = 0, 1, \ldots, q)$ are some complex numbers. In the sequel of this paper we will use rational numbers.

2. Modeling Short Term Forecasting Time Series

2.1. Model with linear fractional recurrence relation of q-th order (LFRq)

For the given time series $x_1, x_2, \ldots, x_n$, to make the analysis simpler, we take $n = i + q$.

The predicted value x_{n+1}, i.e. x_{i+q+1}, applying the method of linear fractional recurrence relation of q-th order, is:

$$x_{i+q+1} = \frac{\alpha_0 + \alpha_1 \cdot x_{i+1} + \alpha_2 \cdot x_{i+2} + \cdots + \alpha_q \cdot x_{i+q}}{\beta_0 + \beta_1 \cdot x_{i+1} + \beta_2 \cdot x_{i+2} + \cdots + \beta_q \cdot x_{i+q}} = \frac{\alpha_0 + \sum\limits_{k=1}^{q} \alpha_k \cdot x_{i+k}}{\beta_0 + \sum\limits_{k=1}^{q} \beta_k \cdot x_{i+k}}.$$

In this model we have $2 \cdot (q+1)$ unknown parameters $(\{\alpha_i, \beta_i\})$, where $(i = 0, 1, \ldots, q)$. Now the question is: what is the best way to determine these parameters? Here the best is in the sense of the most accurate prediction of the value x_{i+q+1}.

The natural way to find these parameters is from the condition that linear fractional recurrence relation of q-th order is correct for $2 \cdot (q+1)$ previous values of time series from the close past, i.e. for values $x_{i-(q+1)}, x_{i-(q+1)+1}$,

$\ldots, x_{i+q}$. In other words, that the equations are true (for $i \geq 2 \cdot (q+1)$):

$$x_{i-(q+1)} = \frac{\alpha_0 + \sum\limits_{k=1}^{q} \alpha_k \cdot x_{i-2\cdot(q+1)+k}}{\beta_0 + \sum\limits_{k=1}^{q} \beta_k \cdot x_{i-2\cdot(q+1)+k}},$$

$$x_{i-(q+1)+1} = \frac{\alpha_0 + \sum\limits_{k=1}^{q} \alpha_k \cdot x_{i-2\cdot(q+1)+k+1}}{\beta_0 + \sum\limits_{k=1}^{q} \beta_k \cdot x_{i-2\cdot(q+1)+k+1}},$$

$$\vdots$$

$$\vdots$$

$$x_{i-(q+1)+2\cdot q+1} = \frac{\alpha_0 + \sum\limits_{k=1}^{q} \alpha_k \cdot x_{i-2\cdot(q+1)+k+2\cdot q+1}}{\beta_0 + \sum\limits_{k=1}^{q} \beta_k \cdot x_{i-2\cdot(q+1)+k+2\cdot q+1}}.$$

From these conditions we get homogeneous system of $2 \cdot (q+1)$ linear equations with $2 \cdot (q+1)$ unknown parameters (α_i, β_i), where $(i = 0, 1, \ldots, q)$:

$$0 = \alpha_0 + \sum_{k=1}^{q} \alpha_k \cdot x_{i-2\cdot(q+1)+k} - x_{i-(q+1)} \cdot \left(\beta_0 + \sum_{k=1}^{q} \beta_k \cdot x_{i-2\cdot(q+1)+k}\right)$$

$$0 = \alpha_0 + \sum_{k=1}^{q} \alpha_k \cdot x_{i-2\cdot(q+1)+k+1} - x_{i-(q+1)+1} \cdot \left(\beta_0 + \sum_{k=1}^{q} \beta_k \cdot x_{i-2\cdot(q+1)+k+1}\right)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$0 = \alpha_0 + \sum_{k=1}^{q} \alpha_k \cdot x_{i-2\cdot(q+1)+k+2\cdot q+1} - x_{i-(q+1)+2\cdot q+1} \cdot \left(\beta_0 + \sum_{k=1}^{q} \beta_k \cdot x_{i-2\cdot(q+1)+k+2\cdot q+1}\right).$$

This system has trivial solution $\alpha_i = \beta_i = 0$, which is useless. Of course, there is a possibility that this system, except trivial, sometimes has also an infinite number of solutions (α_i, β_i), where $(i = 0, 1, \ldots, q)$, but it depends on the values of time series, so we cannot expect, in advance, that this system has another solution besides trivial one. It is interesting to explore what characteristics of time series insures the solutions that are not trivial.

For all the above mentioned, we propose the modification of the model, in the sense that we in advance define value of the parameter $\alpha_0 = A \neq 0$. We call this special linear fractional recurrence relation of q-th order LFRq$(\alpha_0 = A)$.

2.2. Forecasting with a special linear fractional recurrence relation of q-th order LFRq $(\alpha_0 = A)$

The predicted value of time series x_{n+1}, i.e. x_{i+q+1}, applying the method of special linear fractional recurrence relation of q-th order (LFRq $(\alpha_0 = A)$), is:

$$x_{i+q+1} = \frac{A + \alpha_1 \cdot x_{i+1} + \alpha_2 \cdot x_{i+2} + \cdots + \alpha_q \cdot x_{i+q}}{\beta_0 + \beta_1 \cdot x_{i+1} + \beta_2 \cdot x_{i+2} + \cdots + \beta_q \cdot x_{i+q}} = \frac{A + \sum_{k=1}^{q} \alpha_k \cdot x_{i+k}}{\beta_0 + \sum_{k=1}^{q} \beta_k \cdot x_{i+k}}. \tag{2.2.1}$$

In this model we have $2 \cdot q + 1$ unknown parameters $(\beta_0, \alpha_i, \beta_i)$, where $(i = 1, \dots, q)$, and we find them from the condition that modified linear fractional recurrence relation of q-th order is correct for $2 \cdot q + 1$ previous values of time series from the close past, i.e. for the values $x_{i-(q+1)+1}, x_{i-(q+1)+2}, \dots, x_{i+q}$. In other words, that following equations are true (for $i \geq 2 \cdot q + 1$):

$$\begin{aligned}
x_{i-(q+1)+1} &= \frac{A + \sum_{k=1}^{q} \alpha_k \cdot x_{i-2\cdot(q+1)+k+1}}{\beta_0 + \sum_{k=1}^{q} \beta_k \cdot x_{i-2\cdot(q+1)+k+1}}, \\
x_{i-(q+1)+2} &= \frac{A + \sum_{k=1}^{q} \alpha_k \cdot x_{i-2\cdot(q+1)+k+2}}{\beta_0 + \sum_{k=1}^{q} \beta_k \cdot x_{i-2\cdot(q+1)+k+2}}, \\
&\vdots \\
&\vdots \\
x_{i-(q+1)+2\cdot q+1} &= \frac{A + \sum_{k=1}^{q} \alpha_k \cdot x_{i-2\cdot(q+1)+k+2\cdot q+1}}{\beta_0 + \sum_{k=1}^{q} \beta_k \cdot x_{i-2\cdot(q+1)+k+2\cdot q+1}}.
\end{aligned}$$

From these conditions we get the following system of $2 \cdot q + 1$ linear equations with $2 \cdot q + 1$ unknown parameters $(\beta_0, \alpha_i, \beta_i)$, where $(i = 1, \dots, q)$:

$$\begin{aligned}
&\sum_{k=1}^{q} \alpha_k \cdot x_{i-2\cdot(q+1)+k+1} - x_{i-(q+1)+1} \cdot \left(\beta_0 + \sum_{k=1}^{q} \beta_k \cdot x_{i-2\cdot(q+1)+k+1})\right) = -A \\
&\sum_{k=1}^{q} \alpha_k \cdot x_{i-2\cdot(q+1)+k+2} - x_{i-(q+1)+2} \cdot \left(\beta_0 + \sum_{k=1}^{q} \beta_k \cdot x_{i-2\cdot(q+1)+k+2}\right) = -A \qquad (2.2.2) \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
&\sum_{k=1}^{q} \alpha_k \cdot x_{i-2\cdot(q+1)+k+2\cdot q+1} - x_{i-(q+1)+2\cdot q+1} \cdot \left(\beta_0 + \sum_{k=1}^{q} \beta_k \cdot x_{i-2\cdot(q+1)+k+2\cdot q+1}\right) = -A.
\end{aligned}$$

This system has unique solution when the determinant of the system is different from zero, which has place practically always. However, if the determinant of the system is equal to zero, one can do some small modifications in values of time series, which will not affect time series but insure unique solution.

We tested this model on the data from the past (historical simulation) for $\alpha_0 = A = 1$ and $q = 1, 2, 3, 4$.

3. Test of the Method and Data Description

We used the real time series for the forecasting with special linear fractional recurrence relation of q-th order (LFRq $(\alpha_0 = A)$), for $A = 1$ and $q = 1, 2, 3, 4$. We programmed modules in the software Matlab, which made the predictions using the data from the past, based on equations (2.2.1) and (2.2.2) for$A = 1$ and $q = 1, 2, 3, 4$.

We compared the forecasted values with original data from the past, calculated absolute and relative errors of deviation, and in that way analyzed the characteristics of suggested model.

All daily data are retrieved from the site www.euronext.com. For the lack of the space, in the section below we show only the results for the Belgium market, where we used stock price index BEL20, closing price over the period from 1.11.2006 till 30.11.2006.

We also tested the model for each of the 20 companies separately from which BEL20 index is composed, using the values of their stocks (for example, Belgacom, Ackermans & van Haaren, Mobistar, Omega Pharma, etc.). Model was also applied to the other markets, like French and German market, where we used daily data of CAC40 and DAX30 indices, and values of stocks of their companies.

4. Estimation of Results

4.1. Estimation of results for forecasting with LFR1 $(\alpha_0 = 1)$

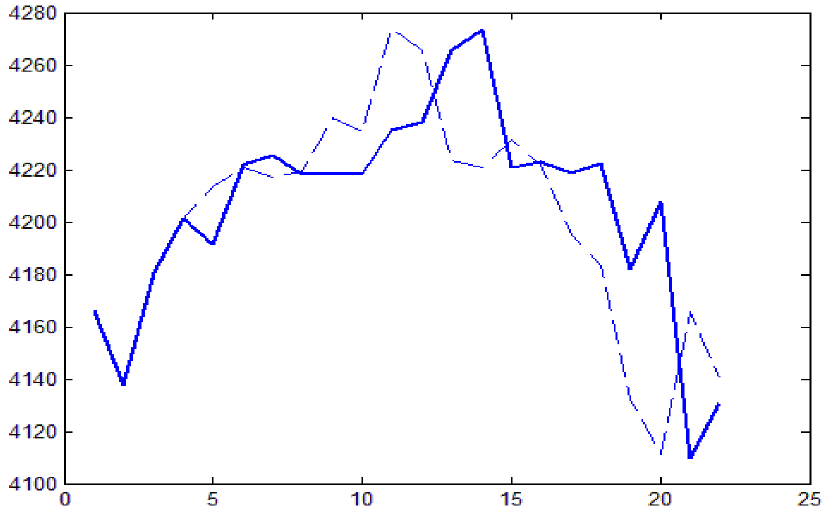
The predicted values from the past $x^{,}_{i+q+1}$, we get using the formula (see (2.2.1)):

$$x^{,}_{i+1+1} = \frac{1+\alpha_1 \cdot x_{i+1}}{\beta_0+\beta_1 \cdot x_{i+1}} \text{ for } i = 3, 4, \ldots n-2,$$

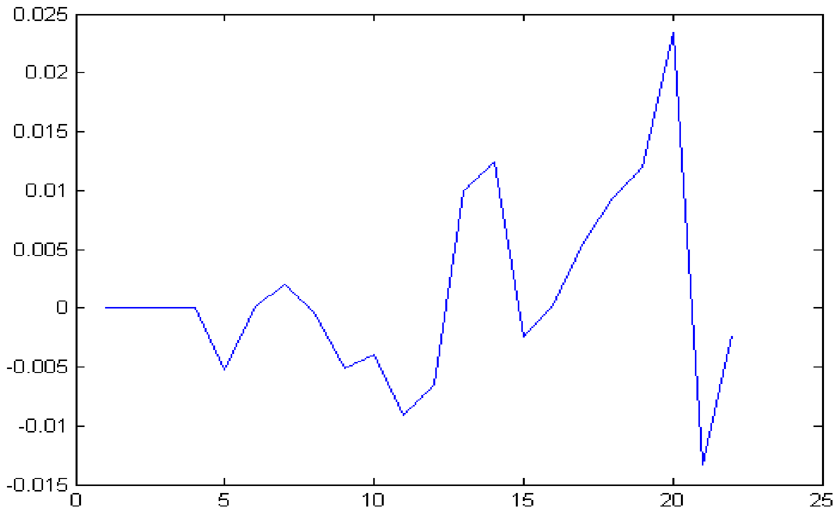
where we get $\alpha_1, \beta_0, \beta_1$ from the (2.2.2).

We then compare predicted data $x^{,}_{i+1+1}$ with the original ones x_{i+1+1}, calculate absolute error, $abs.err. = \left(x^{,}_{i+1+1} - x_{i+1+1}\right)$, and relative error, $rel.err. = \frac{x^{,}_{i+1+1} - x_{i+1+1}}{x_{i+1+1}}$, for $i = 3, 4, \ldots n-2$.

Graphs of predicted and original data, and relative error, for **LFR1** $(\alpha_0 = 1)$ are shown below.



Graph 1: November – real and prognostic process **LFR1** ($\alpha_0 = 1$) (thin broken line – original data, full line – prognostic data).



Graph 2: Relative error **LFR1** ($\alpha_0 = 1$).

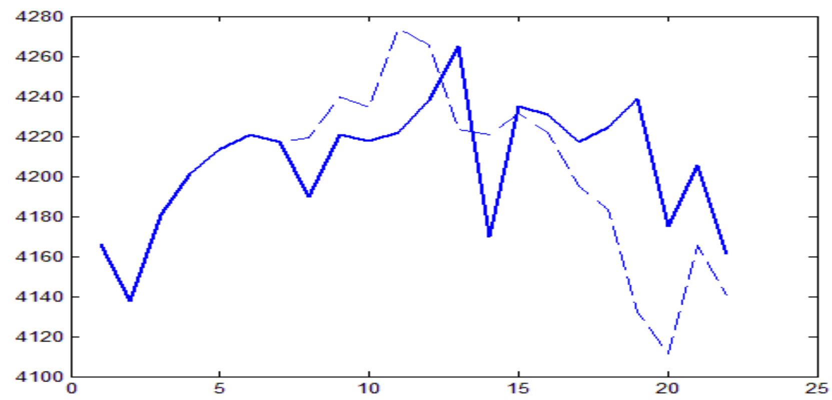
4.2. Estimation of results for forecasting with LFR2 ($\alpha_0 = 1$)

The predicted values from the past x'_{i+q+1}, we get using the formula (see (2.2.1)):

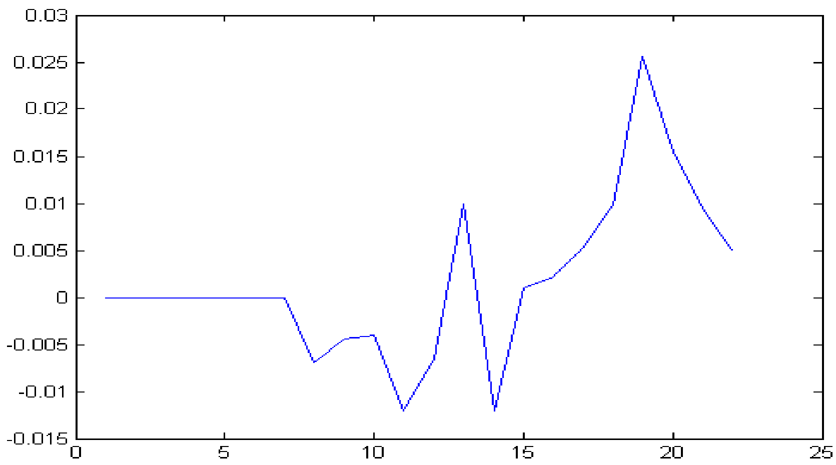
$$x^{,}_{i+2+1} = \frac{1 + \alpha_1 \cdot x_{i+1} + \alpha_2 \cdot x_{i+2}}{\beta_0 + \beta_1 \cdot x_{i+1} + \beta_2 \cdot x_{i+2}} \qquad \text{for } i = 5, 6, \ldots n-3,$$

where we get $\alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2$ from the (2.2.2).
We then compare predicted data $x^{,}_{i+2+1}$, with the original data x_{i+2+1}, calculate absolute error, $abs.err. = \left(x^{,}_{i+2+1} - x_{i+2+1}\right)$, and relative error, $rel.err. = \frac{x^{,}_{i+2+1} - x_{i+2+1}}{x_{i+2+1}}$, for $i = 5, 6, \ldots n-3$.

Graphs of predicted and original data, and relative error, for **LFR2** $(\alpha_0 = 1)$ are presented below.



Graph 3: November – real and prognostic process **LFR2** $(\alpha_0 = 1)$ (thin broken line – original data, full line – prognostic data).



Graph 4: Relative error **LFR2** $(\alpha_0 = 1)$.

4.3. Estimation of results for forecasting with LFR3 ($\alpha_0 = 1$)

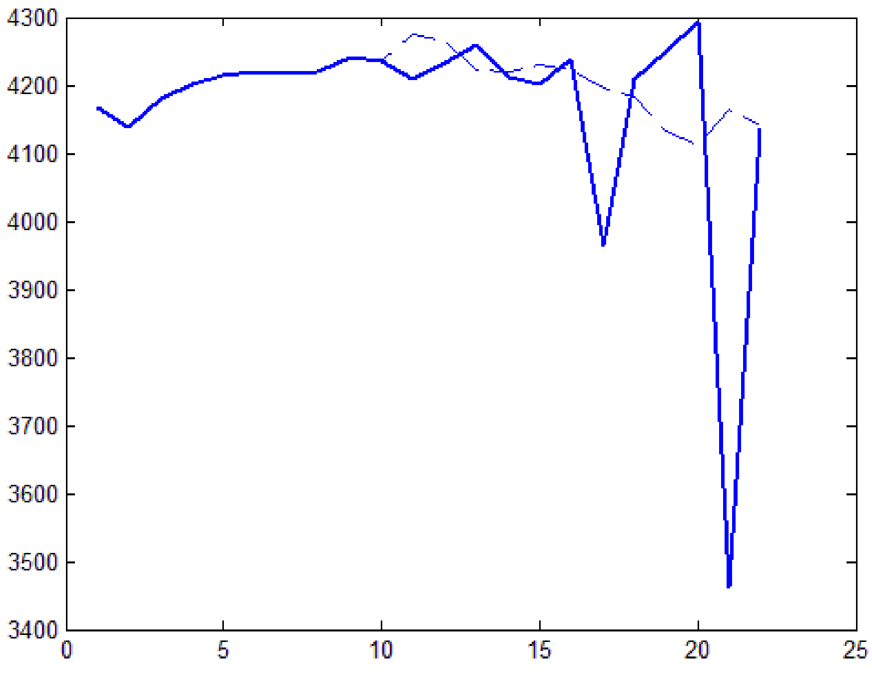
The predicted values from the past $x^{,}_{i+q+1}$, we get using the formula (see (2.2.1)):

$$x^{,}_{i+3+1} = \frac{1 + \alpha_1 \cdot x_{i+1} + \alpha_2 \cdot x_{i+2} + \alpha_3 \cdot x_{i+3}}{\beta_0 + \beta_1 \cdot x_{i+1} + \beta_2 \cdot x_{i+2} + \beta_3 \cdot x_{i+3}} \qquad \text{for } i = 7, 8, \ldots n-4,$$

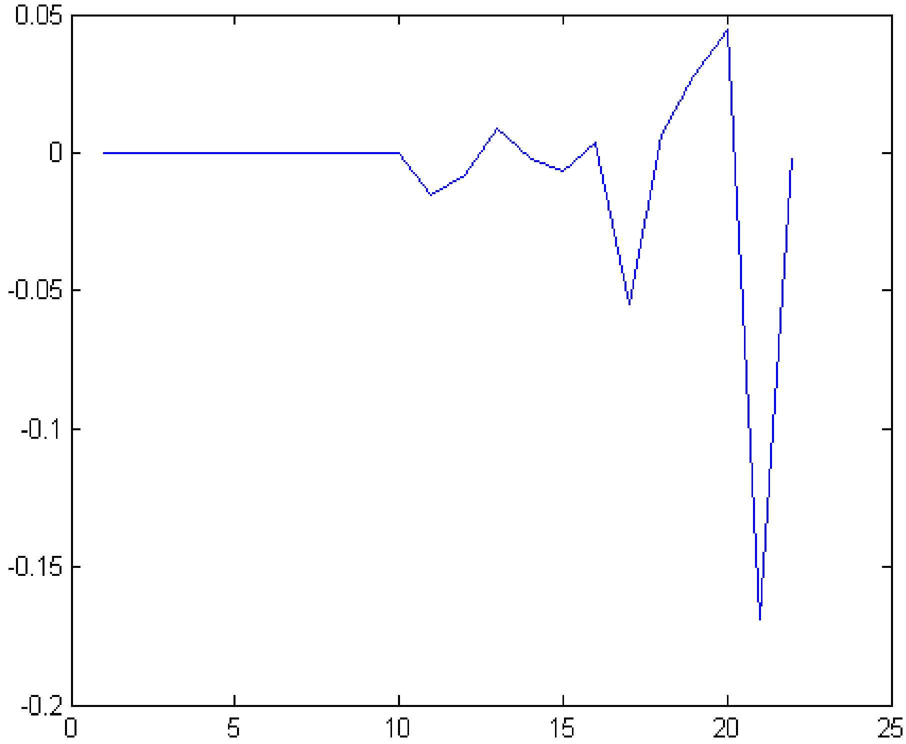
where we have got $\alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3$ from the (2.2.2).

Then we compare predicted data $x^{,}_{i+3+1}$ with the original ones x_{i+3+1}, calculate absolute error, $abs.err. = \left(x^{,}_{i+3+1} - x_{i+3+1}\right)$, and relative error, $rel.err. = \frac{x^{,}_{i+3+1} - x_{i+3+1}}{x_{i+3+1}}$, for $i = 7, 8, \ldots n-4$.

Graphs of predicted and original data, and relative error, for **LFR3**($\alpha_0 = 1$) are shown below.



Graph 5: November – real and prognostic process **LFR3** ($\alpha_0 = 1$) (thin broken line – original data, full line – prognostic data).



Graph 6: Relative error **LFR3** $(\alpha_0 = 1)$.

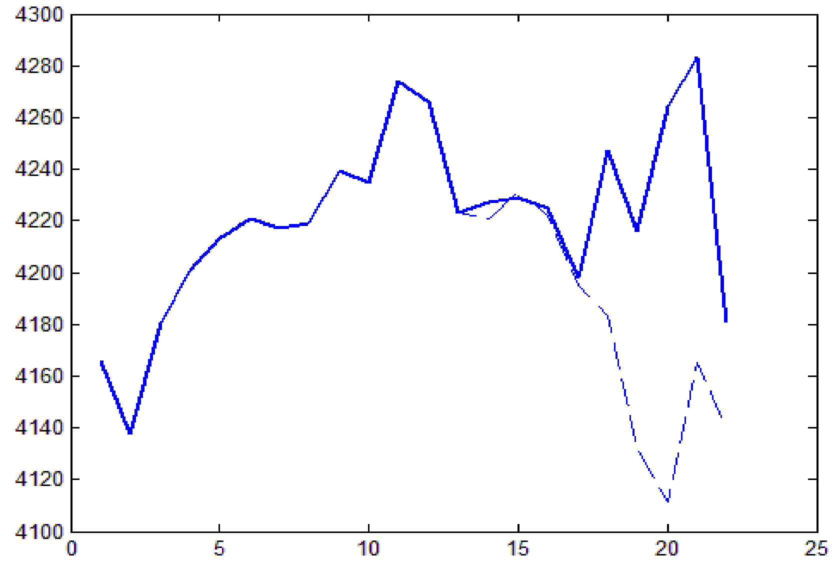
4.4. Estimation of results for forecasting with LFR4 $(\alpha_0 = 1)$

The predicted values from the past $x^{,}_{i+q+1}$, we get using the formula (see (2.2.1)):

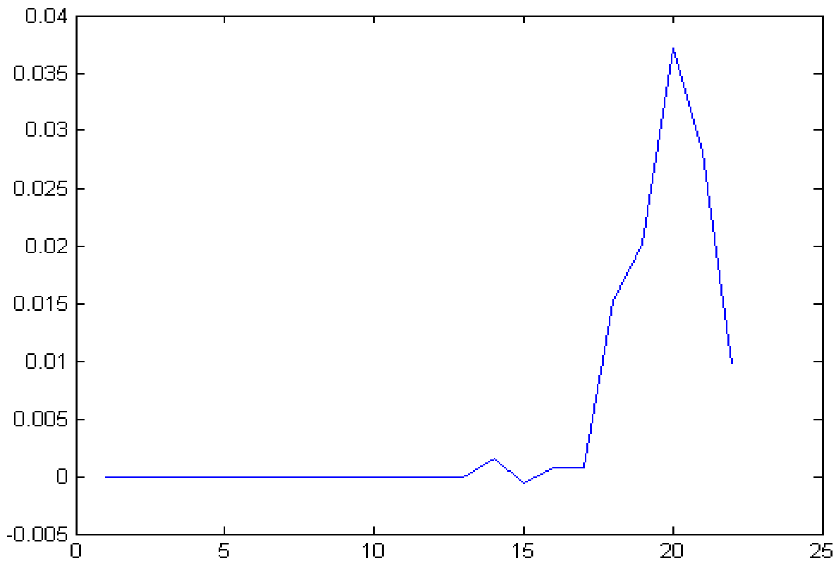
$$x^{,}_{i+4+1} = \frac{1 + \alpha_1 \cdot x_{i+1} + \alpha_2 \cdot x_{i+2} + \alpha_3 \cdot x_{i+3} + \alpha_4 \cdot x_{i+4}}{\beta_0 + \beta_1 \cdot x_{i+1} + \beta_2 \cdot x_{i+2} + \beta_3 \cdot x_{i+3} + \beta_4 \cdot x_{i+4}}$$
$$\text{for } i = 9, 10, \ldots n - 5,$$

where we obtained $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_0, \beta_1, \beta_2, \beta_3, \beta_4$ from the (2.2.2).
We also compare predicted data $x^{,}_{i+4+1}$ with the original ones x_{i+4+1}, calculate absolute error, $abs.err. = \left(x^{,}_{i+4+1} - x_{i+4+1}\right)$, and relative error, $rel.err. = \frac{x^{,}_{i+4+1} - x_{i+4+1}}{x_{i+4+1}}$, for $i = 9, 10, \ldots n - 5$.

Graphs of predicted and original data, and relative error, for **LFR4** ($\alpha_0 = 1$) are presented below.



Graph 7: November – real and prognostic process **LFR4** ($\alpha_0 = 1$) (thin broken line – original data, full line – prognostic data).



Graph 8: Relative error **LFR4** ($\alpha_0 = 1$).

5. Modeling with LFR When the Determinant of the System is Equal to 1

In this part we did some further investigations for the case when the determinant of the system is equal to 1. One can see, from the previous results, that the best predictions are achieved for $q = 1$, so now we have the system which consists of three equations and one condition for the determinant.

5.1. Methodology of modeling with LFR1 (det = 1)

For the given x_i, $i = 1, 2, ..., n$, $(n \geq 4)$, the system becomes:

$$x_2 = \frac{\alpha_0 + \alpha_1 x_1}{\beta_0 + \beta_1 x_1} \quad \Longrightarrow \quad \alpha_0 + \alpha_1 x_1 - \beta_0 x_2 - \beta_1 x_1 x_2 = 0 \tag{1}$$

$$x_3 = \frac{\alpha_0 + \alpha_1 x_2}{\beta_0 + \beta_1 x_2} \quad \Longrightarrow \quad \alpha_0 + \alpha_1 x_2 - \beta_0 x_3 - \beta_1 x_2 x_3 = 0 \tag{2}$$

$$x_4 = \frac{\alpha_0 + \alpha_1 x_3}{\beta_0 + \beta_1 x_3} \quad \Longrightarrow \quad \alpha_0 + \alpha_1 x_3 - \beta_0 x_4 - \beta_1 x_3 x_4 = 0 \tag{3}$$

$$\alpha_0 \beta_1 - \alpha_1 \beta_0 = 1\,. \tag{4}$$

From the (1) we get:

$$\alpha_0 = \beta_0 x_2 + \beta_1 x_1 x_2 - \alpha_1 x_1\,. \qquad (5.1a)$$

Replacing the (5.1a) into the (2) we get:

$$\begin{aligned} \beta_0 x_2 + \beta_1 x_1 x_2 - \alpha_1 x_1 + \alpha_1 x_2 - \beta_0 x_3 - \beta_1 x_2 x_3 &= 0 \\ \alpha_1 (x_2 - x_1) + \beta_0 (x_2 - x_3) + \beta_1 (x_1 x_2 - x_2 x_3) &= 0 \qquad (5.2a) \\ \alpha_1 (x_2 - x_1) + \beta_0 (x_2 - x_3) + \beta_1 x_2 (x_1 - x_3) &= 0\,. \end{aligned}$$

Replacing the (5.1a) into the (3) we get:

$$\begin{aligned} \beta_0 x_2 + \beta_1 x_1 x_2 - \alpha_1 x_1 + \alpha_1 x_3 - \beta_0 x_4 - \beta_1 x_3 x_4 &= 0 \\ \alpha_1 (x_3 - x_1) + \beta_0 (x_2 - x_4) + \beta_1 (x_1 x_2 - x_3 x_4) &= 0\,. \qquad (5.3a) \end{aligned}$$

Replacing the (5.1a) into the (4) we get:

$$\begin{aligned} (\beta_0 x_2 + \beta_1 x_1 x_2 - \alpha_1 x_1)\, \beta_1 - \alpha_1 \beta_0 &= 1 \\ \beta_0 \beta_1 x_2 + \beta_1^2 x_1 x_2 - \alpha_1 \beta_1 x_1 - \alpha_1 \beta_0 &= 1\,. \qquad (5.4a) \end{aligned}$$

Now, we solve the system (5.2a), (5.3a), (5.4a):

$$\begin{aligned} \alpha_1 (x_2 - x_1) + \beta_0 (x_2 - x_3) + \beta_1 x_2 (x_1 - x_3) &= 0 \qquad (5.2a) \\ \alpha_1 (x_3 - x_1) + \beta_0 (x_2 - x_4) + \beta_1 (x_1 x_2 - x_3 x_4) &= 0 \qquad (5.3a) \\ \beta_0 \beta_1 x_2 + \beta_1^2 x_1 x_2 - \alpha_1 \beta_1 x_1 - \alpha_1 \beta_0 &= 1\,. \qquad (5.4a) \end{aligned}$$

From the (5.2a), for $x_2 - x_1 \neq 0$, i.e. for every i, $x_{i+1} \neq x_i$, we get:

$$\alpha_1 = \frac{\beta_0 (x_3 - x_2) + \beta_1 x_2 (x_3 - x_1)}{x_2 - x_1} . \qquad (5.2aa)$$

Replacing (5.2aa) into (5.3a) we get:

$$\frac{\beta_0 (x_3 - x_2) + \beta_1 x_2 (x_3 - x_1)}{x_2 - x_1} (x_3 - x_1) + \beta_0 (x_2 - x_4) + \beta_1 (x_1 x_2 - x_3 x_4) = 0$$

or,

$$\beta_0 \left(\frac{(x_3 - x_2)(x_3 - x_1)}{x_2 - x_1} + (x_2 - x_4) \right) + \beta_1 \left(\frac{x_2 (x_3 - x_1)^2}{x_2 - x_1} + (x_1 x_2 - x_3 x_4) \right) = 0 . \qquad (5.3aa)$$

Replacing (5.2aa) into (5.4a) we get:

$$\beta_0 \beta_1 x_2 + \beta_1^2 x_1 x_2 - \frac{\beta_0 (x_3 - x_2) + \beta_1 x_2 (x_3 - x_1)}{x_2 - x_1} (\beta_1 x_1 + \beta_0) = 1$$

or,

$$\beta_0 \beta_1 \left(x_2 - \frac{x_1 (x_3 - x_2) + x_2 (x_3 - x_1)}{x_2 - x_1} \right) + \beta_1^2 x_1 x_2 \left(1 - \frac{x_3 - x_1}{x_2 - x_1} \right) - \beta_0^2 \frac{x_3 - x_2}{x_2 - x_1} = 1 . \qquad (5.4aa)$$

Now, let us denote in (5.3aa):

$$P = \frac{(x_3 - x_2)(x_3 - x_1)}{x_2 - x_1} + (x_2 - x_4) ,$$

$$Q = \frac{x_2 (x_3 - x_1)^2}{x_2 - x_1} + (x_1 x_2 - x_3 x_4) .$$

Then (5.3aa) becomes:

$$\beta_0 P + \beta_1 Q = 0 . \qquad (5.3aaa)$$

We denote in (5.4aa):

$$M = x_2 - \frac{x_1 (x_3 - x_2) + x_2 (x_3 - x_1)}{x_2 - x_1} , \quad N = x_1 x_2 \left(1 - \frac{x_3 - x_1}{x_2 - x_1} \right) ,$$

$$R = \frac{x_3 - x_2}{x_2 - x_1} .$$

So, (5.4aa) becomes

$$\beta_0 \beta_1 M + \beta_1^2 N - \beta_0^2 R = 1 . \qquad (5.4aaa)$$

Now, the system becomes:

$$P\beta_0 + Q\beta_1 = 0\,, \tag{5.3aaa}$$

$$\beta_0\beta_1 M + \beta_1^2 N - \beta_0^2 R = 1\,. \tag{5.4aaa}$$

Further, from the (5.3aaa) we have:

$$\beta_0 = -\frac{Q}{P}\beta_1\,, \quad (P \neq 0)\,.$$

That is:

$$(x_3 - x_2)\,(x_3 - x_1) \neq (x_4 - x_2)\,(x_2 - x_1)$$

or,

$$\frac{x_3 - x_2}{x_2 - x_1} \neq \frac{x_4 - x_3}{x_3 - x_2}\,.$$

Now, (5.4aaa) becomes:

$$\left(-\frac{Q}{P}\right) M\beta_1^2 + \beta_1^2 N - \left(\frac{Q^2}{P^2}\right) R\beta_1^2 = 1\,.$$

Arranging, we get

$$\beta_1^2 = \frac{P^2}{NP^2 - QPM - Q^2R}\,, \qquad \text{where } NP^2 - QPM - Q^2R \neq 0.$$

So, finally we have

$$\beta_1 = \frac{P}{\sqrt{NP^2 - QPM - Q^2R}}, \quad \text{where } NP^2 - QPM - Q^2R > 0\,.$$

Putting back in the (5.3aaa) we get:

$$\beta_0 = -\frac{Q}{P}\beta_1\,,$$

then in the (5.2a)

$$\alpha_1 = \frac{\beta_0\,(x_3 - x_2) + \beta_1 x_2\,(x_3 - x_1)}{x_2 - x_1}\,,$$

and in the (1)

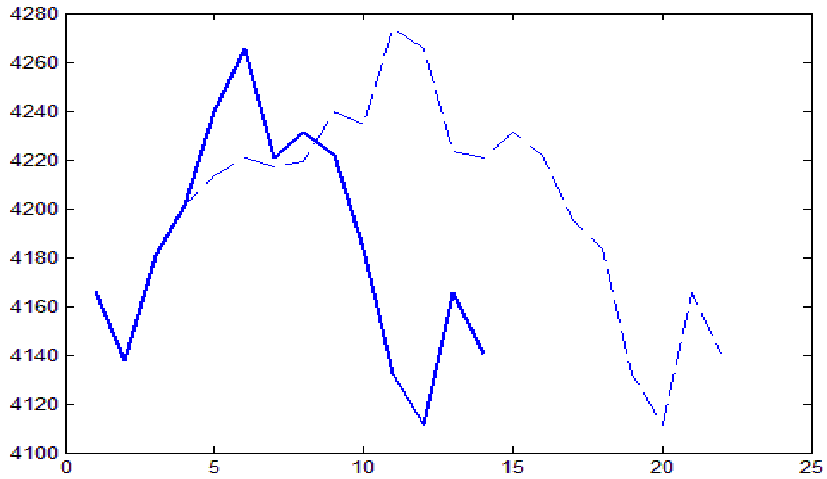
$$\alpha_0 = \beta_0 x_2 + \beta_1 x_1 x_2 - \alpha_1 x_1\,.$$

Now, the prognosis of the value $x_5^{\text{'}}$ is

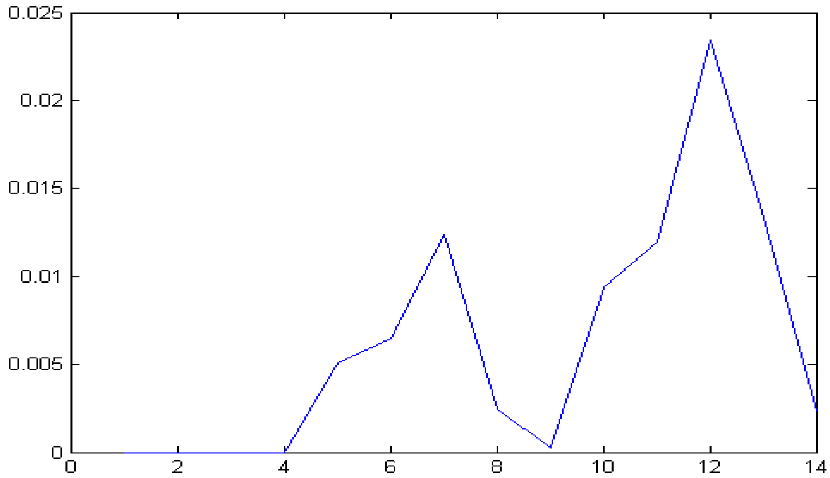
$$x_5^{\text{'}} = \frac{\alpha_0 + \alpha_1 x_4}{\beta_0 + \beta_1 x_4}\,, \qquad \text{where } \alpha_0\beta_1 - \alpha_1\beta_0 = 1.$$

5.2. Testing the theoretical model on the real data

Below are shown the results for Bel20 index for **LFR1**(det = 1), to make analogy with the previous section, for the month November of 2006. The testings were also conducted for the different stock indices, and we also programmed module which made the predictions.



Graph 9: November 2006. – real and prognostic process **LFR1** (det = 1) (thin broken line – original data, full line – prognostic data).



Graph 10: November 2006 - Relative error **LFR1** (det = 1).

6. Concluding Remarks

Comparing the results which we have got for the two cases of modeling, we can conclude that the relative errors for the case of modeling with **LFR**(det = 1), i.e. when the determinant of the system is equal to 1, are significantly smaller. Some details are presented in our works [3, 4].

By Linear Fractional Recurrences financial time series can be considered as dynamical systems not only on the field of real and complex numbers but also on p-adic numbers and adeles (see references [5, 6, 7]).

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