Adelic Quantum Mechanical Hamiltonians and Their Path Integrals*

David Weisbart†
Department of Mathematics, University of California Los Angeles
Los Angeles, CA 90095, USA

Abstract

We discuss in this article some basic features of quantum systems whose configuration space is the ring of rational adeles. We also propose a probabilistic method for specifying the free hamiltonian operators in a quantum dynamical theory over such a configuration space and introduce a natural class of free adelic hamiltonians that generate a measure on an adelic path space.

1. Introduction

It is possible to construct quantum systems over non-Archimedean configuration spaces and such systems were investigated in, for example, [7], [2], [12], [5], [1], [3]. In the case of a hybrid quantum system over the p-adic number field, $\mathbb{Q}_p$, where time is given to be real, the associated hamiltonian operators in [7] and [3] are not differential operators as in the real case but pseudo-differential operators. The free part of these operators give rise to a measure on the Skorohod space, $D[0,\infty)$, of right continuous paths taking values in $\mathbb{Q}_p$ with only jump discontinuities. As shown in [3], for a wide class of matrix potentials it is possible to prove a Feynman-Kac formula for the dynamical semigroup associated to such a system.

In [5], the dynamical operator for an adelic harmonic oscillator is discussed and some interesting formulas are discovered. The dynamical operator is given to be an integral operator but no path integral formulations are attempted. In [4], a Feynman integral is discussed for systems with quadratic lagrangians. The adelic path integral in [4] and the integral operator in [5] are not considered as arising naturally from a differential or pseudo-differential operator as is established in the real and p-adic cases in, for example, [7], [3], [10], [6] and elsewhere. There is no measure that can be

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† e-mail address: dweisbar@math.ucla.edu

rigorously defined on path space that allows the kernel of the dynamical evolution operator given in [5] to be described as integrals over path space and for this reason the integral in [4] is only formal. The kernels of the adelic dynamical operators in [5] and [4] are postulated to be products of kernels given in various p-adic components. It is important to note that time was not considered to be real in [5] and [4] but was instead assumed to be adelic.

In this paper we restrict ourselves to a hybrid system with real time and an adelic configuration space. One issue that was not discussed in the above papers was that of the determination of the free hamiltonian operators. In particular, given a real, p-adic, or adelic quantum system, is it possible in a natural way to determine the free hamiltonian operators such that they are unique up to some "mass" parameter? We discuss this question in Section 3. Another question that arises in considering the papers [4] and [5] is why one should consider a dynamical operator corresponding to a free particle as a product of the various real and p-adic dynamical operators. In Section 4, we construct a family of adelic Hamiltonian operators and show that these operators generate a measure on an adelic path space. This suggests that it is possible to have a Feynman-Kac formula in this setting that yields a formula for the dynamical semigroups analogous in form to the dynamical groups discussed in [5] in that both are products over real and p-adic components.

2. On the Dynamics of a Quantum Mechanical Theory

A basic postulate of quantum theory is that time evolution is given by a unitary representation, U, of $\mathbb{R}$ on the quantum Hilbert space, $\mathcal{H}$, so that for each $t \in \mathbb{R}$, $U : t \mapsto U_t : \mathcal{H} \to \mathcal{H}$, $U_0 = I$ and $U_{t+s} = U_t U_s$. A state, $\phi \in \mathcal{H}$ at time $t = 0$ evolves to the state $U_t\phi$ in time t. By Stone's Theorem, there is a densely defined self-adjoint operator, H, such that $(U_t\phi)(x) = (e^{-itH}\phi)(x)$ for each t. H is called the hamiltonian of the system. The choice of a hamiltonian completely describes the dynamics of a quantum theory. In many specific situations, a real quantum hamiltonian can be constructed formally from a classical hamiltonian that describes an analogous classical system by making a formal replacement of the position and momentum variables in the classical hamiltonian by the position and momentum operators. The question in general is, how does one determine the hamiltonian operators in a dynamical quantum mechanical theory given a kinematical system? We shall try and attempt an answer to this question although, unfortunately, at the present time the discussion is somewhat speculative.

General Considerations Without any further requirement, there are many possibilities for hamiltonian operators. We specialize to a subclass of such operators that are commonly known as *free hamiltonians*, that is, hamiltonian operators corresponding to non-interacting particles.

In general, we consider the configuration space to be a locally compact second countable self-dual abelian group, X. We fix on X the Haar measure,

μ. Since X is self dual, we denote X^* by X as well making the appropriate identifications and denote by $\langle \cdot, \cdot \rangle \colon X \times X \to S^1$ the dual pairing. The map $\mathcal{F}$ is given by:

$$(\mathcal{F}f)(x) = \int_K \langle -x, \xi \rangle f(\xi)\, d\xi\,, \quad f \in L^1(X)\,.$$

We scale the Haar measure so that it is self-dual in the sense that the inverse Fourier transform $\mathcal{F}^{-1}$ is given by:

$$(\mathcal{F}^{-1}f)(x) = \int_K \langle x, \xi \rangle f(\xi)\, d\xi\,, \quad f \in L^1(X)\,.$$

$\mathcal{F}$ is an isometry with respect to the L^2-norm and can be extended to a unitary operator, the Fourier transform, on all of $L^2(X)$. The quantum Hilbert space with configuration space X is given to be $L^2(X)$ with respect to the self-dual Haar measure. Denote by $[\cdot, \cdot]$ the commutator acting on pairs of operators on $\mathcal{H}$. We postulate the following:

Postulate 1: Let G be a group of symmetries acting on X and T is a representation of G on $L^2(G)$. For each $g \in G$, $[T_g, H] = 0$.

This condition was discussed in [8] and the relationship is to be understood in the sense that the spectral measures of T_g and H commute.

Postulate 2: A free hamiltonian operator generates a stochastic semigroup with independent increments giving rise to a measure on the Skorohod space, $D([0, \infty); X)$, of right continuous paths on X.

Imposition of these conditions would seem a reasonable way to determine quantum hamiltonians and we will see that these conditions indeed allow some degree of specification.

The Real Setting The real setting with configuration space $X = \mathbb{R}^3$ was discussed in depth in [8]. If one only considers that time evolution is given by a unitary representation of $\mathbb{R}$ on the quantum Hilbert space, then by Stone's theorem, time evolution is given by e^{itH} where H is a self-adjoint operator. Of course, this in no way serves to specify H. Consider then the projective representations on the quantum Hilbert space of the covering group of $\mathcal{G}$ where $\mathcal{G}$ is the group of space-time symmetries, the 10 dimensional Galilean group. We assume that time evolution is a unitary representation of $\mathbb{R}$ that lifts to one of these projective representations of the covering group of $\mathcal{G}$. In [8], V. S. Varadarajan classifies all of the projective representations of the covering group of $\mathcal{G}$. Once this is done, all of the possible unitary representations of $\mathbb{R}$ that lift in the required way are known. Up to a mass parameter, there is only one: a constant times the laplace operator.

We arrive at a similar result by appealing to Postulates 1 and 2 above, although a further assumption appears necessary. An argument given in [8] shows that Postulate 1 implies that the free hamiltonian is necessarily the fourier transform of multiplication by a borel function of the laplace operator. If the path space in Postulate 2 is to be the space of continuous paths taking values in $\mathbb{R}^n$, then the only function of the laplace operator that generates a measure on this space is given by a positive constant times the laplace operator and so we have the desired result that the only quantum mechanical Hamiltonian operators acting on $L^2(\mathbb{R}^n)$ satisfying both Postulates 1 and 2 are given by $H = k\Delta$ where k is a positive real parameter. Forcing paths to be continuous does not seem to be a natural assumption and e^{-tH} is a stochastic semigroup when H is the fourier transform of multiplication by $|x|^b$ so long as $0 < b \leq 2$. It would be interesting to know if different values of b give rise to measures on the Skorohod space as required by Postulate 2. We will investigate this at a later time.

The p-Adic Setting For each prime p, the p-adic numbers, denoted $\mathbb{Q}_p$, are the completion of the rational numbers under the p-adic valuation. We assume the reader has a basic familiarity with p-adic analysis and we cite [12] as a reference. $\mathbb{Q}_p$ is a local field and a locally compact second countable abelian group under addition and therefore we can treat quantum systems on $\mathbb{Q}_p$ as discussed above.

Denote by $\mathbb{Z}_p$ the ring of integers in $\mathbb{Q}_p$, $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \colon |x| \leq 1\}$. If χ_p is a non-trivial character on $(\mathbb{Q}_p, +)$, then any other character ψ on $(\mathbb{Q}_p, +)$ is of the form $\psi(x) = \chi_p(ax)$ for some $a \in \mathbb{Q}_p$. Let dx_p denote normalized Haar measure on $(\mathbb{Q}_p, +)$, i.e.,

$$\int_{\mathbb{Z}_p} dx_p = 1\,.$$

Pick a rank zero character χ_p on $(\mathbb{Q}_p, +)$, and define the Fourier transform $\mathcal{F}_p$ by:

$$\hat{f}(x) = (\mathcal{F}_p f)(x) = \int_{\mathbb{Q}_p} \chi_p(-x\xi) f(\xi)\, d\xi\,, \quad f \in L^1(\mathbb{Q}_p)$$

properly normalized so that the measure is self-dul as discussed above. Of course, $\mathcal{F}_p$ is an isometry with respect to the L^2-norm and can be extended to a unitary operator on all of $L^2(\mathbb{Q}_p)$.

Since $\mathbb{Q}_p$ is a second countable locally compact abelian group, we may appeal to above arguments in the real case. In particular, χ_p is an additive character and so $\chi_p((x + y)(z)) = \chi_p(xy)\chi_p(yz)$. Thus, defining $K_p = \mathcal{F}_p H_p \mathcal{F}_p^{-1}$ with H_p the p-adic hamiltonian, Postulate 1 implies in the p-adic case as well that K_p is given as multiplication by a function of the p-adic valuation. The argument is essentially the same as in the real setting. In fact, just as in the real case for f belonging to a sufficient densely defined class of functions, $(K_p f)(x) = k(|x|_p) f(x)$ for some Borel function k. In this case it turns out that Postulate 2 does not narrow down the choices of the function k very much. In [7] it shown that if $k(x) = |x|_p^b$ where $b > 0$ then H generates a family of path measures associated to the stochastic semi-group e^{-tH_p}. It would be interesting to find more general classes of functions of the norm that give rise to measures on the Skorohod space. Since the continuous paths on $\mathbb{Q}_p$ are constant paths, we cannot make a further restriction as is possible in the real setting. The Hamiltonians in the p-adic case are, therefore, not uniquely defined given the restrictions imposed by the two postulates. The case when the configuration space is $\mathbb{Q}_p^n$ may be handled in a similar way.

3. The Adelic Setting

Adelic configuration space. Adelic configuration space, $\mathbb{A}$, is the restricted direct product of $\mathbb{R}$ and $\mathbb{Q}_p$ for each prime p. This is to say that if $a \in \mathbb{A}$ then $a = (a_\infty, a_2, a_3, \dots)$ where $a_p \notin \mathbb{Z}_p$ for at most finitely many p. The adelic norm $|\cdot|_{\mathbb{A}}$ is defied in the following way:

$$|a|_{\mathbb{A}} = |a_\infty| \times \prod_p |a_p|_p, \qquad a = (a_\infty, a_2, a_3, \dots)$$

where $|\cdot|_p$ is the norm on each p-adic component. $\mathbb{A}$ is a self-dual, second countable, locally compact abelian group and hence may be treated by the same general principles as discussed above. A character on $\mathbb{A}$, $\chi_{\mathbb{A}}$, is a map $a \mapsto \chi_\infty(a_\infty)\chi_2(a_2)\cdots\chi_p(a_p)\cdots$. Denote by Ω_p the characteristic function on $\mathbb{Z}_p$. A function is *simple* if $f(a) = f_\infty(a_\infty)f_2(a_2)f_3(a_3)\cdots f_p(a_p)\cdots$ where $f_p \neq \Omega_p$ for only finitely many p, each f_p is Schwarz-Bruhat and f_∞ is infinitely differentiable with compact support. The set of Schwarz-Bruhat functions on $\mathbb{A}$, $\mathcal{S}_{\mathbb{A}}$, is the collection of all finite sums of simple functions on $\mathbb{A}$. Let $dx_{\mathbb{A}}$ be the self-dual Haar measure on $\mathbb{A}$. Then $\mathcal{S}_{\mathbb{A}}$ is dense in $L^2(\mathbb{A}, dx_{\mathbb{A}})$. The fourier transform of a simple function, f, is defined in the following way:

$$\begin{aligned}(\mathcal{F}_{\mathbb{A}}f)(x) &= \int_{\mathbb{A}} \chi_{\mathbb{A}}(-xy)f(y)\, dy_{\mathbb{A}} \\ &= \int_{\mathbb{R}} \chi_\infty(-x_\infty y_\infty)f_\infty(x_\infty)\, dx_\infty \prod_p \int_{\mathbb{Q}_p} \chi_p(-x_p y_p) f(y_p)\, dy_p.\end{aligned}$$

With $dx_{\mathbb{A}}$ appropriately scaled, the fourier transform is an isometric map from $\mathcal{S}_{\mathbb{A}}$ to $L^2(\mathbb{A}, dx_{\mathbb{A}})$ and extends to an automorphism of $L^2(\mathbb{A}, dx_{\mathbb{A}})$.

Adelic hilbert space. Although it is natural to choose $L^2(\mathbb{A})$ to be the adelic hilbert space, there is more freedom in choosing the adelic hilbert space than in choosing the real or p-adic hilbert spaces due to the richer structure of the adeles. In particular, it is possible to consider adelic fock spaces. We denote by $\mathcal{H}_p = L^2(\mathbb{Q}_p)$ for p prime and $H_p = L^2(\mathbb{R})$ for $p = \infty$. Let $\mathcal{V} = (v_\infty, v_2, v_3, v_5, \ldots)$ be a sequence of distinguished unit vectors with $v_p \in \mathcal{H}_p$ for each prime p and for $p = \infty$. Given such a distinguished sequence, we can form the von Neumann tensor product over all p, $\mathcal{H}_{\mathbb{A},\mathcal{V}}$, of the hilbert spaces $\mathcal{H}_p$. Simple functions are functions of the form $f_\infty \otimes f_2 \otimes f_3 \otimes \cdots$ where $f_p \neq v_p$ for at most finitely many p, f_p is a Schwarz-Bruhat function for each prime p and f_∞ is smooth and compactly supported. The pre-fock space, $\hat{\mathcal{H}}_{\mathbb{A},\mathcal{V}}$, is then the set of finite sums of simple functions and has a norm given to be the product of the norms on each component. The fock space, $\mathcal{H}_{\mathbb{A},\mathcal{V}}$, is the completion of the pre-fock space under the given norm. For any $\mathcal{V}$, there is an isometric isomorphism between $\mathcal{H}_{\mathbb{A}}$ and $\mathcal{H}_{\mathbb{A},\mathcal{V}}$. Given a sequence of operators H_p, each which have a ground state S_p with $H_pS_p = 0$ and $\mathcal{V} = (S_\infty, S_2, S_3, \dots)$, there is a natural operator, H, on $\mathcal{H}_{\mathbb{A},\mathcal{V}}$ defined in the following way. For any simple function f, $Hf = H_\infty f_\infty \otimes f_2 \otimes f_3 + f_\infty \otimes H_2 f_2 \otimes f_3 \cdots$. This sum is a finite sum for any simple function and so the operator is defined on the pre-fock space given that each H_p is defined on the Schwarz-Bruhat functions in $\mathcal{H}_p$.

Determining adelic hamiltonians We let $\mathbb{A}$ be the configuration space of a quantum system. If the symmetry group on $\mathbb{A}$ is generated by the translations by elements of $\mathbb{A}$ and the norm preserving maps, then by the same arguments as in the real and p-adic cases, $K = {\mathcal{F}_{\mathbb{A}}}^{-1} H_{\mathbb{A}} \mathcal{F}_{\mathbb{A}}$ is given by multiplication by some function k of the adelic norm, that is, $(Kf)(a) = k(|a|_{\mathbb{A}})f(a)$. By Postulate 2, $H_{\mathbb{A}}$ must generate a measure on $D([0,\infty);\mathbb{A})$. It is then necessary to determine which operators generate measures on $D([0,\infty);\mathbb{A})$. Although we shall study this at a future time it seems unlikely that this process will allow us a unique determination of the adelic hamiltonian operators. It will be interesting to see how large a class of hamiltonians are possible and the hope is that the study will yield operators that have mathematical and even physical significance. We should note the dynamical operator associated to the hamiltonian operator given above cannot be a tensor product of dynamical operators on each p-adic component.

There are some natural operators that we can investigate over the adeles that give rise to dynamical operators with product structures and we will restrict ourselves to one such operator. Consider the von Neumann tensor product of the various L^2 spaces on each of the p-adics and $\mathbb{Q}_\infty = \mathbb{R}$ where in this case the distinguished vector in each of the Hilbert spaces $L^2(\mathbb{Q}_p)$ is Ω_p, the characteristic function on $\mathbb{Z}_p$. We call this fock space $\mathcal{H}_{\Omega,\mathbb{A}}$. It is possible to identify $L^2(\mathbb{A})$ with this von Neumann tensor product since there is an isometric isomorphism between them. We consider the hamiltonian operator of the form

$$H_{\mathbb{A},\Omega} = H_{\mathbb{R}} \otimes I_2 \otimes I_3 \otimes \cdots + I_\infty \otimes H_2 \otimes I_3 \otimes \cdots + I_\infty \otimes I_2 \otimes H_3 \otimes \cdots + \cdots .$$

Here, H_∞ is the usual real Hamiltonian and H_p are the various p-adic Hamiltonians and all are given by fourier transforms of multiplication operators given by appropriate functions of the p-adic norms. For the rest of this article we will consider the case when, up to a mass parameter, m_p

with $\sum_p m_p < \infty$, $H_p = m_p \mathcal{F}^{-1} M_{|x_p|_p^2} \mathcal{F}$. The condition that $\sum_p m_p < \infty$ ensures that $H_{\mathbb{A},\Omega}$ is a well-defined operator from $\hat{\mathcal{H}}_{\mathbb{A},\Omega}$ to $\mathcal{H}_{\mathbb{A},\Omega}$. Furthermore, with the appropriate identifications, this operator is well defined on the Schwarz-Bruhat functions on $L^2(\mathbb{A})$ and so is densely defined. We name this operator acting on $L^2(\mathbb{A})$ $H_{\mathbb{A}}$. We should remark that there is no canonical choice of a fock space for this operator since there is no ground state for H_p for any p. Thus, without further discussion the choice of the particular fock space must be considered to be artificial.

The Measures Associated to an Adelic Theory For each prime p and also for $p = \infty$, each H_p generates a stochastic semigroup $(\pi_{p,t})_{t>0}$ such that for each $\phi_p \in L^2(\mathbb{Q}_p)$, $(\pi_{p,t}\phi_p)(x) = (e^{-itH_p}\phi_p)(x)$ and associated probability density functions, $f_{p,t}$, such that $(\pi_{p,t}\phi_p)(x) = (e^{-itH_p}\phi_p)(x) = (f_{p,t} * \phi_p)(x)$ where $*$ denotes the convolution. Furthermore, for each $x_{0,p} \in \mathbb{Q}_p$, $f_{p,t}$ gives rise to a pre-measure on the cylinder sets of the path space and then an outer measure on the space of all paths on $\mathbb{Q}_p$ for each p including $p = \infty$. Denote by $\mathcal{O}_p = \{\omega_p \colon [0,\infty) \to \mathbb{Q}_p\}$. Given a finite sequence of time points $\{\tau_0 = 0, \tau_1, \ldots, \tau_k\}$ and a corresponding sequence of Borel subsets of $\mathbb{Q}_p$, $\{B_{p,1}, \ldots, B_{p,k}\}$, a cylinder set $C_{x_{0,p},\tau_0,\ldots,\tau_k,B_{p,1},\ldots,B_{p,k}} = \{\omega_p \in \mathcal{O}_p \mid \omega_p(\tau_0) = x_{0,p},\ \omega_p(\tau_1) \in B_{p,1},\ \ldots,\ \omega_p(\tau_k) \in B_{p,k}\}$ has measure

$$\begin{aligned}
&\mu_{p,x_0}(C_{x_{0,p},\tau_0,\ldots,\tau_k,B_{p,1}\ldots,B_{p,k}}) \\
&\qquad = \int_{B_{p,1}} \cdots \int_{B_{p,k}} f_{p,\tau_1}(x_1 - x_0) f_{p,\tau_2-\tau_1}(x_2 - x_1) \\
&\qquad\qquad\qquad\qquad \cdots f_{p,\tau_k-\tau_{k-1}}(x_k - x_{k-1})\, dx_{k,p} \cdots dx_{1,p}.
\end{aligned}$$

By the arguments in [7], for each prime p, the above measure gives rise to a measure on the σ-algebra generated by the cylinder sets and this measure can be realized as the measure associated to a stochastic process with independent increments on $D_p[0,\infty)$. In the case of $p = \infty$, the measure can be realized as the measure associated to the Wiener process on the space of continuous paths on $\mathbb{R} = \mathbb{Q}_\infty$, $C[0,\infty)$ [6].

Define the path spaces

$$\hat{\mathcal{O}}_{\mathbb{A}} = C[0,\infty) \prod_p D_p[0,\infty) \quad \text{and} \quad \mathcal{O}_{\mathbb{A}} = \{\omega_{\mathbb{A}} \colon [0,\infty) \to \mathbb{A} \mid \omega_{\mathbb{A}} \in \hat{\mathcal{O}}_{\mathbb{A}}\}.$$

Theorem 1 *There is a measure, $\mu_{\mathbb{A}}$, on $\mathcal{O}_{\mathbb{A}}$ induced from the product measure on $\hat{\mathcal{O}}_{\mathbb{A}}$ and*

$$(e^{-tH_{\mathbb{A}}}\phi)(x) = \int_{\mathcal{O}_{\mathbb{A}}} \phi(\omega_{\mathbb{A}}(t))\, d\mu(\omega_{\mathbb{A}}).$$

Furthermore, for any simple function $\phi_{\mathbb{A}}$ *and any* $a \in \mathbb{A}$,

$$\begin{aligned}(e^{-tH_{\mathbb{A}}}\phi)(a) &= \int_{\mathbb{A}} k_{\mathbb{A}}(a, x_{\mathbb{A}})\phi_{\mathbb{A}}(x_{\mathbb{A}})\, dx_{\mathbb{A}} \\ &= \int_{\mathbb{R}} k_\infty(a_\infty, x_\infty)\phi_\infty(x_\infty)\, dx_\infty \prod_p \int_{\mathbb{Q}_p} k_p(a_p, x_p)\phi_p(x_p)\, dx_p,\end{aligned}$$

and for each $p = \infty,\ 2,\ 3,\ 5,\ \dots,$ $k_p(x, y)$ *can be written as an integral over a space of paths on* $\mathbb{R}$, $\mathbb{Q}_2$, $\mathbb{Q}_3$, $\mathbb{Q}_5$, $\dots$ *respectively, and* $k_{\mathbb{A}}(x, y)$ *can be written as an integral over* $\mathcal{O}_{\mathbb{A}}$.

Proof: We sketch the idea of the proof here. A complete proof to a more general result as well as some other computations may be found in an upcoming paper. We first show that the operators $H_{\mathbb{A}}$ is essentially self-adjoint on the space of Schwarz-Bruhat functions on $L^2(\mathbb{A})$ and that $H_{\mathbb{A},\Omega}$ is essentially self-adjoint on the pre-fock space $\hat{\mathcal{H}}_{\mathbb{A},\Omega}$. The main idea of the proof of the theorem is that the measures on $C[0, \infty)$ and on each of the path spaces $D_p[0, \infty)$ induce, for each $a = (a_\infty, a_2, a_3, \dots)$, a measure on the infinite product of the path spaces given by

$$\begin{aligned}P_a(\{\omega(t) = (\omega_\infty(t), \omega_2(t), \cdots) \mid \omega_p(t_i) \in B_{p,i}\ i = 1, \dots, k \text{ and } \omega(0) = a\} \\ = \prod_{p=\infty, p \text{ prime}} P_{a_p}(\{\omega_p(t_i) \in B_{p,i}\ i = 1, \dots, k\})\end{aligned}$$

where for each i, p, $B_{p,i}$ are Borel subsets of $\mathbb{Q}_p$. It is possible to show that $\mathcal{O}_{\mathbb{A}}$ is a Borel subset of $\hat{\mathcal{O}}_{\mathbb{A}}$ and that $\mathcal{O}_{\mathbb{A}}$ has full measure. Thus, the measure on $\hat{\mathcal{O}}_{\mathbb{A}}$ descends to a measure on the adelic path space. It is then possible, given the above measures, to prove the Feynman-Kac formula in the free case given above by employing arguments similar to those used in [10]. We will discuss the case with non-trivial potentials at a later time. □

4. Conclusions

It seems only natural to look for a general principle that allows us to specify quantum hamiltonians. In the real quantum mechanical setting in 3 dimensions, this can be done by requiring that the hamiltonian operators are the infinitesimal generators of the dynamical groups given by unitary representations of $\mathbb{R}$ that lift to projective representations of the 10 dimensional Galilean group. Specification is also possible in any real configuration space by appealing to Postulates 1 and 2 above with the requirement that the path space in Postulate 2 is the space of continuous paths. For quantum dynamical systems in real time over the p-adic numbers, such a specification is not possible and it seems unlikely as well in the adelic case. It is certainly possible to consider dynamics in p-adic time via a representation of $\mathbb{Q}_p$ on $L^2(Q_p^n)$. One should study an analog to the Galilean group over such spaces and the projective representations of such a group in order to specify the dynamics of the system. Such a study has already been undertaken in the recent works of V. S. Varadarajan and Jukka Virtanen [9], [11]. Of course, in this setting, the dynamical operator will not arise from a hamiltonian.

We considered an operator of the form

$$H_{\mathbb{A}} = H_{\mathbb{R}} \otimes I_2 \otimes I_3 \otimes \cdots + I_\infty \otimes H_2 \otimes I_3 \otimes \cdots + I_\infty \otimes I_2 \otimes H_3 \otimes \cdots + \cdots .$$

One should look into the measures that arise from more general adelic hamiltonians and investigate when such measures descend from the space of products of real and p-adic paths to adelic paths to see if there is a general criteria for when this is possible. Furthermore, the von Neumann tensor product of the Hilbert spaces can be built using different distinguished vectors than those we have discussed. One should study path measures over these adelic spaces as well. We shall consider such issues in an upcoming paper as well as give complete proofs to the statements made in this article.

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