# Universal Teichmüller Space

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Abstract

The universal Teichmüller space $\mathcal{T}$ is the space of quasisymmetric homeomorphisms of the unit circle $S^1$, normalized modulo Möbius transformations. It can be realized as an open subset in the complex Banach space of holomorphic quadratic differentials in the unit disc. In this paper we propose a method of quantization of $\mathcal{T}$, based on approach, due to Connes.

The universal Teichmüller space $\mathcal{T}$ consists of quasisymmetric homeomorphisms of the unit circle $S^1$ (i.e. homeomorphisms of $S^1$, extending to quasiconformal maps of the unit disc $\Delta$), considered modulo Möbius transformations. This space, introduced by Ahlfors and Bers, plays a key role in the theory of quasiconformal maps and Riemann surfaces. It has a natural complex structure, generated by embedding $\mathcal{T}$ into the complex Banach space of holomorphic quadratic differentials in $\Delta$. The space $\mathcal{T}$ contains all classical Teichmüller spaces $T(G)$, where $G$ is a Fuchsian group, as complex submanifolds. On the other hand, the space $\mathcal{S} := \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$ of diffeomorphisms of the circle, normalized modulo Möbius transformations, may be considered as a "smooth" part of $\mathcal{T}$.

According to Nag–Sullivan [4], there is a natural action of the group $\mathrm{QS}(S^1)$ of quasisymmetric homeomorphisms of $S^1$ on the Sobolev space $V := H_0^{1/2}(S^1, \mathbb{R})$ of half-differentiable functions on $S^1$. Moreover, this action is symplectic with respect to a natural symplectic form $\omega$ on $V$. By this action, the universal Teichmüller space $\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$ can be identified with a space of complex structures on $V$, compatible with $\omega$.

We propose a method of quantization of $\mathcal{T}$, based on approach, due to Connes. Though the described $\mathrm{QS}(S^1)$-action on $\mathcal{T}$ cannot be differentiated in classical sense (in particular, there is no Lie algebra, associated to $\mathrm{QS}(S^1)$), we can define a quantized infinitesimal version of this action. Namely, there is a quantum differential $d^q h$, associated with any quasisymmetric homeomorphism $h \in \mathrm{QS}(S^1)$. This differential is an integral operator on the Sobolev space $V$ with kernel, given by the finite-difference

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derivative of $h$. In these terms the quantization of $\mathcal{T}$ is given by the quantum algebra of derivations of Fock space $F(V,J)$, generated by quantum differentials $d^q h$.

## 1. UNIVERSAL TEICHMÜLLER SPACE

A homeomorphism $f : S^1 \to S^1$ is called *quasisymmetric* if it can be extended to a quasiconformal homeomorphism $w$ of the unit disc $\Delta$. It means that $w$ has locally integrable derivatives and satisfies the *Beltrami equation*

$$w_{\bar{z}} = \mu(z) w_z$$

for almost all $z \in \Delta$. Here, $\mu \in L^\infty(\Delta)$ is a function with $\|\mu\|_\infty < 1$, called the *Beltrami differential* of $w$.

Orientation-preserving quasisymmetric homeomorphisms of $S^1$ form a group $\mathrm{QS}(S^1)$ with respect to composition. Since any orientation-preserving diffeomorphism $f \in \mathrm{Diff}_+(S^1)$ extends to a diffeomorphism of the closed unit disc $\overline{\Delta}$, which is evidently quasiconformal, we have $\mathrm{Diff}_+(S^1) \subset \mathrm{QS}(S^1)$.

The quotient space

$$\mathcal{T} := \mathrm{QS}(S^1)/\text{Möb}(S^1) \ ,$$

where $\text{Möb}(S^1)$ denotes the Möbius group of fractional-linear automorphisms of the unit disc $\Delta$, restricted to $S^1$, is called the *universal Teichmüller space*. It can be identified with the space of normalized quasisymmetric homeomorphisms of $S^1$, fixing the points $\pm 1$ and $-i$.

As we have pointed out, there is an inclusion

$$\mathcal{S} := \mathrm{Diff}_+(S^1)/\text{Möb}(S^1) \hookrightarrow \mathcal{T} = \mathrm{QS}(S^1)/\text{Möb}(S^1)$$

of the space $\mathcal{S}$ of normalized diffeomorphisms of $S^1$ into the universal Teichmüller space. We consider $\mathcal{S}$ as a smooth part of $\mathcal{T}$.

The term "universal" in the name of the universal Teichmüller space is due to the fact that $\mathcal{T}$ contains all classical Teichmüller spaces $T(G)$, where $G$ is a Fuchsian group, as complex submanifolds. If a Riemann surface $X$ is uniformized by the unit disc $\Delta$, so that $X = \Delta/G$, then the corresponding Techmüller space $T(G)$ may be identified with the quotient

$$T(G) = \mathrm{QS}(S^1)^G/\text{Möb}(S^1) \ ,$$

where $\mathrm{QS}(S^1)^G$ denotes the subgroup of $G$-invariant quasisymmetric homeomorphisms in $\mathrm{QS}(S^1)$.

We can also define $\mathcal{T}$ directly in terms of Beltrami differentials. The space $B(\Delta)$ of all Beltrami differentials in the unit disc $\Delta$ can be identified (as a set) with the unit ball in the complex Banach space $L^\infty(\Delta)$. Given a Beltrami differential $\mu \in B(\Delta)$, we can extend it to a Beltrami differential $\check{\mu}$ on the extended complex plane $\overline{\mathbb{C}}$ by setting it equal to zero outside $\Delta$. Then, applying the existence theorem for quasiconformal maps on the complex plane (cf. [1]), we find a normalized quasiconformal homeomorphism $w^\mu$, satisfying the Beltrami equation on $\overline{\mathbb{C}}$ with Beltrami differential $\check{\mu}$. This homeomorphism is conformal on the exterior $\Delta_-$ of the closed unit disc $\overline{\Delta}$ on $\overline{\mathbb{C}}$ and fixes the points $\pm 1, -i$. Introduce an equivalence relation between Beltrami differentials in $\Delta$: two Beltrami differentials $\mu$ and $\nu$ are equivalent if $w^\mu|_{\Delta_-} \equiv w^\nu|_{\Delta_-}$. Then the universal Teichmüller space $\mathcal{T}$ can be identified with the quotient

$$\mathcal{T} = B(\Delta)/\sim$$

of the space $B(\Delta)$ of Beltrami differentials modulo introduced equivalence relation.

We introduce a complex structure on $\mathcal{T}$, using its embedding into the space of quadratic differentials. Namely, given an arbitrary point $[\mu]$ of $\mathcal{T}$, represented by a Beltrami differential $\mu \in B(\Delta)$, we associate with it the Schwarz derivative $S(w^\mu|_{\Delta_-})$ of the conformal map $w^\mu|_{\Delta_-}$. Due to invariance of Schwarzian under Möbius transformations, the image of $\mu$ under this map depends only on the class $[\mu]$ of $\mu$ in $\mathcal{T}$, and defines a holomorphic quadratic differential in $\Delta_-$. (The latter fact follows from the transformation properties of Beltrami differentials, prescribed by the Beltrami equation.) Composing the defined map with a fractional-linear biholomorphism of $\Delta_-$ onto $\Delta$, we obtain an embedding

$$\Psi : \mathcal{T} \longrightarrow B_2(\Delta) \ , \quad [\mu] \longmapsto \psi(\mu) \ ,$$

associating with a point $[\mu]$ of the universal Teichmüller space $\mathcal{T}$ a holomorphic quadratic differential $\psi(\mu)$ in $\Delta$.

The space $B_2(\Delta)$ of holomorphic quadratic differentials in $\Delta$ is a complex Banach space with respect to the natural hyperbolic norm, given by

$$\|\psi\|_2 := \sup_{z\in\Delta}(1-|z|^2)^2|\psi(z)|$$

for a quadratic differential $\psi$.

The constructed map $\Psi : \mathcal{T} \to B_2(\Delta)$, called *Bers embedding*, is a homeomorphism of $\mathcal{T}$ onto an open bounded connected contractible subset in $B_2(\Delta)$ (cf. [3]).

Using the constructed embedding, we can introduce a complex structure on the universal Teichmüller space $\mathcal{T}$ by pulling it back from the complex Banach space $B_2(\Delta)$. It provides $\mathcal{T}$ with the structure of a complex Banach manifold. The composition of the natural projection

$$B(\Delta) \longrightarrow \mathcal{T} = B(\Delta)/\sim$$

with the constructed map $\Psi$ defines a holomorphic map

$$F : B(\Delta) \longrightarrow B_2(\Delta)$$

of complex Banach manifolds.

## 2. SOBOLEV SPACE OF HALF-DIFFERENTIABLE FUNCTIONS

The Sobolev space of half-differentiable functions on $S^1$ is a Hilbert space $V := H_0^{1/2}(S^1,\mathbb{R})$, consisting of functions $f \in L^2(S^1,\mathbb{R})$ with zero average over the circle, having generalized derivatives of order 1/2 in $L^2(S^1,\mathbb{R})$. In terms of Fourier series, a function $f \in L^2(S^1,\mathbb{R})$ with Fourier series

$$f(z) = \sum_{k\neq 0} f_k z^k \ , \quad f_k = \bar{f}_{-k} \ , \quad z = e^{i\theta} \ ,$$

belongs to $H_0^{1/2}(S^1,\mathbb{R})$ if and only if it has a finite Sobolev norm of order 1/2:

$$\|f\|_{1/2}^2 = \sum_{k\neq 0} |k||f_k|^2 = 2\sum_{k>0} k|f_k|^2 < \infty \ .$$

The space $V$ can be provided with a natural symplectic structure, given by a 2-form $\omega : V \times V \to \mathbb{R}$, defined in terms of Fourier coefficients of $\xi, \eta \in V$ by

$$\omega(\xi,\eta) = 2\mathrm{Im} \sum_{k>0} k\xi_k\bar{\eta}_k \ .$$

It has also a complex structure $J$, given in terms of Fourier decompositions by

$$\xi(z) = \sum_{k\neq 0} \xi_k z^k \longmapsto (J\xi)(z) = -i\sum_{k>0} \xi_k z^k + i\sum_{k<0} \xi_k z^k \ .$$

This complex structure is compatible with symplectic form $\omega$ and, in particular, defines a Kähler metric on $V$. In other words, $V$ is a Kähler Hilbert space.

The complex structure operator $J$ determines a decomposition of the complexified vector space $V^{\mathbb{C}}$ into the direct sum

$$V^{\mathbb{C}} = W_+ \oplus W_- = W \oplus \overline{W} \ ,$$

where $W_\pm$ is the $(\mp i)$-eigenspace of $J \in \mathrm{End}\, V^{\mathbb{C}}$. The subspaces $W_\pm$ are isotropic with respect to symplectic form $\omega$ and the splitting $V^{\mathbb{C}} = W_+ \oplus W_-$ is an orthogonal direct sum with respect to the Kähler metric.

With any orientation-preserving homeomorphism $f$ of $S^1$ we can associate an operator $T_f : L^2(S^1,\mathbb{R}) \to L^2(S^1,\mathbb{R})$, given by change of variable:

$$T_f(\xi) := \xi \circ f - \frac{1}{2\pi}\int_0^{2\pi} \xi\,(f(\theta))\,d\theta\ .$$

According to Nag-Sullivan [4], the operator $T_f$ acts on $V$, i.e. $T_f : V \to V$, if and only if $f \in \mathrm{QS}(S^1)$. Moreover, for any $f \in \mathrm{QS}(S^1)$ the operator $T_f$ acts on $V$ by symplectic transformations, preserving the form $\omega$. Its complex-linear extension to $V^{\mathbb{C}}$ preserves the holomorphic subspace $W$ if and only if $f \in \mathrm{Möb}(S^1)$, and in this case $T_f$ is a unitary operator on $W$.

It follows that quasisymmetric homeomorphisms act on the Sobolev space $V$ by bounded symplectic operators. Hence, we have an embedding

$$\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1) \longrightarrow \mathrm{Sp}(V)/\mathrm{U}(W)\ ,$$

where $\mathrm{Sp}(V)$ denotes the symplectic group of $V$ and $\mathrm{U}(W)$ is a unitary group of $W$. The space $\mathrm{Sp}(V)/\mathrm{U}(W)$ can be identified with the space of complex structures on $V$, compatible with $\omega$.

## 3. QUANTIZATION OF $\mathcal{T}$

We shall use the following definition of quantization, due to Connes [2]. Suppose that the algebra of observables $\mathfrak{A}$ is an associative involutive algebra of functions, provided with an exterior differential $d$. Its quantization is a representation $\pi$ of $\mathfrak{A}$ in a Hilbert space $H$, sending the differential $df$ of a function $f \in \mathfrak{A}$ into the commutator $[S,\pi(f)]$ of the operator $\pi(f)$ with a self-adjoint symmetry operator $S$ with $S^2 = I$. The differential here is understood in the sense of non-commutative geometry, i.e. as a linear map $d : \mathfrak{A} \to \Omega^1(\mathfrak{A})$, satisfying the Leibnitz rule (cf. [2]). In other words, the quantization is a representation of the algebra $\mathrm{Der}(\mathfrak{A})$ of derivations of $\mathfrak{A}$ in the Lie algebra $\mathrm{End}\,H$. (Recall that a derivation of an algebra $\mathfrak{A}$ is a linear map: $\mathfrak{A} \to \mathfrak{A}$, satisfying the Leibnitz rule. Derivations of an algebra $\mathfrak{A}$ form a Lie algebra, since the commutator of two derivations is again a derivation.)

If the algebra of observables $\mathcal{A}$ contains non-smooth functions, the differential $df$ of a non-smooth observable $f \in \mathfrak{A}$ is not defined in the classical sense, but its quantum analogue $d^q f$, given by

$$d^q f := [S,\pi(f)]\ ,$$

may still have sense, as it is demonstrated by the following example.

Denote by $\mathfrak{A}$ the algebra $L^\infty(S^1,\mathbb{C})$ of bounded functions on the circle $S^1$. Any function $f \in \mathfrak{A}$ defines a bounded multiplication operator in the

Hilbert space $H = L^2(S^1, \mathbb{C})$:

$$M_f : h \in H \longmapsto fh \in H \ .$$

The operator $S$ is given by the Hilbert transform $S : L^2(S^1, \mathbb{C}) \to L^2(S^1, \mathbb{C})$:

$$(Sf)(e^{i\varphi}) = \frac{1}{2\pi} V.P. \int_0^{2\pi} K(\varphi, \psi) f(e^{i\psi}) d\psi \ ,$$

where the integral is taken in the principal value sense, and $K(\varphi, \psi)$ is the Hilbert kernel

$$K(\varphi, \psi) = 1 - i \cot \frac{\varphi - \psi}{2} \ .$$

The differential $df$ of a general observable $f \in \mathfrak{A}$ is not defined in the classical sense, but its quantum analogue

$$d^q f := [S, M_f]$$

is defined as an operator in $H$.

We apply these ideas to the quantization of $\mathcal{T}$. To simplify the formulas, we switch from the unit circle $S^1$ to the real line $\mathbb{R}$, replacing the Sobolev space $V = H_0^{1/2}(S^1, \mathbb{R})$ by its counter-part $V_R := H^{1/2}(\mathbb{R}, \mathbb{R}) \equiv H^{1/2}(\mathbb{R})$ on the real line. The operator $S$ is given again by the Hilbert transform

$$(Sf)(s) = \frac{1}{\pi i} \text{V.P.} \int \frac{f(t)}{s - t} \, dt \ , \quad f \in L^2(\mathbb{R}) \ .$$

The quantum differential $d^q f = [S, M_f]$ of a function $f \in L^\infty(\mathbb{R})$ is an operator on $L^2(\mathbb{R})$, given by

$$d^q f(h) = \frac{1}{\pi i} \int k(s, t) h(t) \, dt \tag{1}$$

with the kernel, equal to

$$k(s, t) = \frac{f(s) - f(t)}{s - t} \ , \quad s, t \in \mathbb{R} \ .$$

Note that the quasiclassical limit of this operator, defined by taking the value of the kernel on the diagonal (i.e. by taking the limit for $s \to t$), coincides with the multiplication operator $h \mapsto f'h$, so the quantization reduces in this case to the replacement of derivative by its finite-difference analogue.

In section 2. we have defined a natural action of quasisymmetric homeomorphisms on $V_R$. This action does not admit the differentiation, so there is no Lie algebra, associated to $\mathrm{QS}(\mathbb{R})$. In other words, there is no classical algebra of observables, associated to $V_R$. The situation is similar to that, considered in the example, and, as in example, we can still define a corresponding quantum object.

For that we extend the $\mathrm{QS}(\mathbb{R})$-action on $V_R$ to symmetry operators by setting

$$S^h := h \circ S \circ h^{-1} \tag{2}$$

for $h \in \mathrm{QS}(\mathbb{R})$. The quantized infinitesimal version of (2) is given by the integral operator $d^q h : V_R \to V_R$, defined by (1). We can extend this operator to the Fock space $F(W)$ by defining it first on the basis elements of $F(W)$ by Leibnitz rule, and then extending by linearity to all finite elements of $F(W)$. The completion of this operator yields an unbounded operator $d^q h$ on $F(W)$. Note that this defines a quantum counter-part of the algebra of observables, which itself is not defined. However, we can identify it with the quantum derivation algebra, generated by extended operators $d^q h$ with $h \in \mathrm{QS}(\mathbb{R})$.

Summing up, the Connes quantization of the universal Teichmüller space $\mathcal{T}$ is performed in two stages. The "first quantization" consists of defining a quantized infinitesimal version of the $\mathrm{QS}(\mathbb{R})$-action on symmetry operators. More precisely, we consider the $\mathrm{QS}(\mathbb{R})$-action (2) on symmetry operators, as an action on $\mathcal{T}$, identified with a space of compatible complex structures on $V_R$. Its quantized infinitesimal version is given by quantum differentials $d^q h = [S, M_h]$. The "second quantization" is performed by the extension of quantum differentials $d^q h$ to the Fock space $F(W)$. The extended operators $d^q h$ with $h \in \mathrm{QS}(\mathbb{R})$ generate the corresponding quantum derivation algebra on $F(W)$.

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