Arithmetic gravity and its adelic properties

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Abstract

This talk is a survey lecture on parts of the dissertation thesis of the author. The complete thesis is available at arXiv:hep-th/0809.3579. We present an approach to adelic physics based on the language of algebraic spaces. Relative algebraic spaces $X \to S$ are considered as fundamental objects which describe space-time. This yields a number field invariant formulation of general relativity which, in the special case $S = \operatorname{Spec} \mathbb{C}$, may be translated back into the language of manifolds. With regard to adelic physics the choice of S as an excellent Dedekind scheme is of interest (e.g. $S = \operatorname{Spec} \mathbb{Z}$). In this arithmetic case, it turns out that X is a Néron model. This enables us to make concrete statements concerning the structure of the space-time described by X.

1. Preface

Unless otherwise specified, let $K \subset \mathbb{R}$ be an algebraic number field (i.e. a finite algebraic extension of $\mathbb{Q}$), and let $\mathcal{O}_K$ be the ring of integral numbers of K (i.e. the integral closure of $\mathbb{Z}$ in K). For example, think of $\mathcal{O}_K = \mathbb{Z}$ and $K = \mathbb{Q}$.

Since 1987, there have been many interesting applications of p-adic numbers in physics. In his influential paper [9], I.V. Volovich draws the vision of number theory as the ultimate physical theory, where numbers are proposed as the fundamental entities of the universe. It is argued that the development of physics over arbitrary (number) fields might be necessary. In particular, this implies the incorporation of p-adic numbers in physical theories. Since then, many p-adic and even *adelic* models have been constructed (see for example [8] for an overall view). Adeles enable us to regard real and p-adic numbers simultaneously. More precisely, an adele is an infinite tuple

$$x = (x_2, \dots, x_p, \dots, x_\infty),$$

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where $x_\infty \in \mathbb{R}$ and $x_p \in \mathbb{Q}_p$ with the restriction that one has $x_p \in \mathbb{Z}_p$ for all but a finite set of primes. In a certain way, these adelic models unify the ordinary (i.e. $\mathbb{R}$-valued) and p-adic models.

Adelic models of gravity are the starting point of this thesis. But, instead of working directly with adeles and the respective adelic space-time models as it is usually done, we will study a new, purely geometric approach to adelic physics based on relative algebraic spaces $X \to S$, $S = \operatorname{Spec} \mathcal{O}_K$. However, there are close relations between these two approaches as it may be seen in the following example.

0.1 Example. Let us choose $K = \mathbb{Q}$. Consequently, $\mathcal{O}_K = \mathbb{Z}$ and $S = \operatorname{Spec} \mathbb{Z}$. Furthermore assume that the relative algebraic space X over S is representable by a smooth S-scheme, i.e. let us consider a smooth morphism $\pi : X \to \operatorname{Spec} \mathbb{Z}$ of schemes. Set-theoretically, $\operatorname{Spec} \mathbb{Z}$ consists of infinitely many closed points (one point for each prime number p) plus one generic point which we will denote by ∞, and which corresponds to the zero ideal of $\mathbb{Z}$. Furthermore, X may be viewed as union $\bigcup_p \pi^{-1}(p) \cup \pi^{-1}(\infty)$ of the fibres of π.

In our arithmetic setting (and in analogy to complex algebraic geometry), a “physical point” x is given by an S-valued point of X, i.e. by a section $s : \operatorname{Spec} \mathbb{Z} \hookrightarrow X$ of the structure morphism π (i.e. $\pi \circ s = \mathrm{id}$). More precisely, x is given by the image of s. But, set-theoretically, s cuts out one closed point in each fibre. Thus, in analogy to the adelic situation, an S-valued point x may be viewed as a set of points:

$$x = \{x_2, \ldots, x_p, \ldots, x_\infty\}.$$

Furthermore, according to the point of view of adelic physics, each archimedean point (resp. each morphism over the archimedean prime spot at infinity) is only the archimedean component of an adelic point (resp. an adelic morphism). In short, everything in the archimedean world comes from the adelic level. If now $\varphi : Y \to \operatorname{Spec} \mathbb{Z}$ is an arbitrary smooth S-scheme and if we denote by Y_K the pre-image $\varphi^{-1}(\infty)$ of ∞ under φ, the above extension property (from the archimedean to the adelic level) reads as follows in algebraic geometry:

For every K-morphism $f_K : Y_K \to X_K$, there is an S-morphism $f : Y \to X$ which extends f_K. $(\star)$

All in all, instead of adeles, the set $X(S)$ of S-valued points of an algebraic space $X \to S$ is the set of interest in our approach. The objective of this thesis is the investigation of a new approach to general relativity and (pure) Yang-Mills theory based on algebraic spaces. The condition $(\star)$ makes clear why Néron models will be of particular interest.

2. Introduction

According to the theory of general relativity, space-time may be described by means of a differentiable manifold. Thereby, gravity is encoded in a metrical tensor g which satisfies the Einstein equations. More precisely, our starting point are the complex gravitational field equations. Then, any solution of the Einstein equations gives rise to a complex manifold. For technical reasons, we will once and for all assume that this classical space-time manifold may be realized as a compact complex manifold $\mathfrak{X}$ which is *Moishezon*. The latter condition means that

$$\operatorname{transdeg}_{\mathbb{C}}\Big(K(\mathfrak{X})\Big) = \dim_{\mathbb{C}} \mathfrak{X},$$

where $K(\mathfrak{X})$ denotes the field of meromorphic functions on $\mathfrak{X}$. For example, all algebraic manifolds fulfill this equation. Therefore, following the ideas of [1], where it is among other things argued that one should restrict to algebraic manifolds in quantum cosmology, our assumption is not too restrictive. However, let us at least mention that there are Moishezon manifolds which are not algebraic. The technical reason why we restrict attention to Moishezon manifolds is the following beautiful theorem due to Artin.

0.2 Theorem. There is an equivalence of categories

$$\Big(\text{Moishezon manifolds}\Big) \leftrightsquigarrow \Big(\text{smooth, proper algebraic spaces over } \mathbb{C}\Big)$$

This theorem enables us to consider the ordinary complex space-time manifold $\mathfrak{X}$ as a complex algebraic space. Now the following observation is crucial. While, on the level of manifolds, the theory is essentially adapted to the complex numbers, the language of algebraic spaces offers to possibility to replace $\mathbb{C}$ by any commutative ring.

In 1987, I.V. Volovich suggested that a fundamental physical theory should be formulated in such a way that it is invariant under change of the underlying number field (see [9]). This motivates the following program which will be studied within the first part of this thesis:

1. Replace the pair $(\mathfrak{X}, g)$ consisting of a (complex) manifold $\mathfrak{X}$ and a metric g by a pair

   $$\Big(X \to S, g\Big),$$

   where X is a smooth, separated algebraic space over a base S, and where g is a metric over X (see Appendix).
2. Starting from exactly the same physical principles as in the realm of manifolds, deduce the equations of Einstein's theory of general relativity in the setting of algebraic spaces over an arbitrary base S (thus realizing a number field invariant theory). Determine the pair $(X \to S, g)$ in such a way that Einstein's equations are fulfilled.
3. Investigate properties of hypothetical space-time models $(X \to S, g)$ depending on the choice of the base S.

0.3 Remark. Principally, there are many interesting possible choices for S. For example, there is the case of positive characteristic, i.e. S might be chosen as the spectrum of a (finite) field or as function field of an algebraic curve over a finite field. However, in those models $(X \to S, g)$, which will be studied within the bounds of this thesis, we will often choose S to be representable by an excellent Dedekind-scheme with field of fractions K of characteristic zero. Then the following two cases are of interest:

1. S is Zariski zero-dimensional and given by the spectrum $\operatorname{Spec} K$ of a field K of characteristic zero. Especially in the case $K = \mathbb{C}$, everything may be translated back into the language of manifolds (by Theorem 0.2).
2. S is Zariski one-dimensional. In this case we are interested in the choice $S = \operatorname{Spec} \mathcal{O}_K$, where $\mathcal{O}_K \subset K$ is the ring of integral numbers of an algebraic number field K (e.g. $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$).

But what is the physics behind the choice $S = \operatorname{Spec} \mathcal{O}_K$? Why should we consider number fields instead of real or complex numbers? Following the ideas of B. Dragovich, V.S. Vladimirov, I.V. Volovich and many others (see e.g. [1] - [6], [8], [9]), let us state at least two arguments at this place. The first argument concerns the process of measurement. While it is not clear at all whether transcendental numbers can be the result of a measurement, integral (or rational) numbers can. Second, we know from Einstein that gravity is encoded in deformations of space-time scales (described by means of the metrical tensor g). Looking at the energy scale that we experience, it is an empiric fact that we may assume that gravity is completely encoded in the archimedean scale and that non-archimedean, p-adic scales may be neglected. Nevertheless, there is no reason why this should be true on all energy scales down to the Planck scale. It is an appealing project to study physical models where not only the ordinary, archimedean degrees of freedom are taken into consideration, but also the p-adic, non-archimedean degrees of freedom. Physically, the adelic approach means:

There is one degree of freedom per primespot and dimension. $(*)$

As already indicated in Example 0.1, the principle $(*)$ may as well be realized by considering algebraic spaces over $\mathcal{O}_K$. This motivates the following Definition 0.4 (whose physical motivation will be illustrated in Remark 0.6). Recall that, given two relative algebraic spaces $X \to S$ and $Y \to S$, we denote by $X(Y)$ the set of S-morphisms $Y \to X$. Furthermore recall that for an algebraic space $\pi : X \to S$ we denotes the fibre of π over the generic point of S by X_K (physically this generic fibre represents the archimedean component of the algebraic space).

0.4 Definition. Let S be an excellent Dedekind scheme with field of fractions K of characteristic zero. Consider a pair $(X \to S, g)$ consisting of:

- a smooth, separated algebraic space $\pi : X \to S$ over S
- a metric g on X (see Appendix)

such that the following conditions are fulfilled:

(i) g satisfies the Einstein equations (see Appendix).

(ii) For each smooth algebraic space $Y \to S$ and each K-morphism $u_K : Y_K \to X_K$ there is an S-morphism $u : Y \to X$ extending u_K.

Then the pair $(X \to S, g)$ is called a *model of type (GR)*.

0.5 Corollary. In the setting of Definition 0.4, let us assume that the algebraic space $\pi : X \to S$ is representable by a smooth and separated S-scheme. Then, the morphism u in Definition 0.4 b) is uniquely determined. In other words, $X \to S$ is the *Néron model* of its generic fibre X_K (see [BLR], Def. 1.2/1). In particular, the following statements hold:

1. If u_K is an isomorphism so is u.
2. For each étale S-scheme S' with field of fractions K' the canonical map $X(S') \to X_K(K')$ is bijective.

PROOF In order to prove the uniqueness assertion let us choose two morphisms u, v extending u_K. Using the separatedness of $X \to S$ we conclude from [7], Prop. 3.3.11, that u and v are equal if they coincide on a dense subset of Y. Therefore, it suffices to show that the generic fibre Y_K of Y is dense in Y. This may be done as follows: Due to smoothness, the structure morphism $f : Y \to S$ is an open map of topological spaces (use [BLR], Prop. 2.4/8 and [EGA IV$_2$], 2.4.6). The openness of f implies that the pre-image $f^{-1}(D)$ of any dense subset D of S is dense in Y. As the generic point of S is dense in S we are done. Consequently, $X \to S$ is the Néron model of its generic fibre.

The statements a) and b) follow directly from the universal property of Néron models. For example, choose $Y = S'$ in order to see b).

0.6 Remark. If $S = \operatorname{Spec} K$ is the spectrum of a field K, condition (ii) of Definition 0.4 is empty. If furthermore $K = \mathbb{R}$, any model of type (GR) induces a solution of Einstein's theory of general relativity (by evaluation at $\mathbb{R}$-valued points). This explains the label *model of type (GR)*, because (GR) shall remind of general relativity. However, in the case $S = \operatorname{Spec} \mathcal{O}_K$ we arrive at the following physical interpretation:

- Condition (ii) implements the "adelic" point of view.

  In order to see this, let us choose $S = \operatorname{Spec} \mathbb{Z}$ and therefore $K = \mathbb{Q}$. Recall that the generic fibre X_K of X represents the archimedean component. Then condition (ii) says that the archimedean world is only the projection from the "adelic" level to the archimedean component. In truth, everything is defined over all prime spots, and there is one degree of freedom per prime spot.

- We saw in Corallary 0.5 that condition (ii) implies a canonical bijection $X_K(K) = X(S)$. Recall that $X_K(K)$ is the set of archimedean points, and that $X(S)$ is the set of "adelic" points. In the special case $K = \mathbb{Q}$, the bijection $X_K(K) \cong X(S)$ means exactly that every archimedean point $x_\infty \in X_K(K)$ of X is in truth only the archimedean element x_∞ of an infinite set of points $x = \{x_2, \ldots, x_p, \ldots, x_\infty\} \in X(S)$. Finally,

> Corallary 0.5 a) reflects the physically crucial statement that any "deformation" of the archimedean component by means of isomorphisms extends to the "adelic" level.

Furthermore, we immediately obtain the interesting result that the pair $(X \to S, g)$ cannot be the flat Minkowski space-time if we are in the "adelic" situation $S = \operatorname{Spec} \mathcal{O}_K$.

PROOF Let $S = \operatorname{Spec} \mathcal{O}_K$ and assume that $(X \to S, g)$ describes the flat, topologically trivial Minkowski space-time. Then

- $g = \operatorname{diag}(\pm 1, \pm 1, \pm 1, \pm 1)$ and
- $X = \mathbb{A}^n_S$ or $X = \mathbb{P}^n_S$ depending on whether we work projective or not.[1]

But if $X = \mathbb{A}^n_S$, then $K^n \cong X_K(K) \neq X(S) \cong \mathcal{O}_K^n$, and if $X = \mathbb{P}^n_S$, not every morphism u_K extends to an S-morphism u (see [BLR], Example 3.5/5).

Therefore, the flat, topologically trivial Minkowski space-time is no longer a vacuum solution in the arithmetic situation. However, there is a vacuum solution with non-trivial topology: One can prove that the fibred product

$$X := E_0 \times_S \ldots \times_S E_n$$

of smooth elliptic curves E_i over S (provided with the first fundamental form g as metric) is a model of type (GR). Like Minkowski space-time, also the product of elliptic curves carries a commutative group structure. This motivates the following definition.

0.7 Definition. Let $(X \to S, g)$ be a model of type (GR) in the sense of Definition 0.4, and let X_K be the generic fibre of X. Then the pair $(X \to S, g)$ is called a *model of type (SR)*, if in addition the following condition holds:

(iii) X_K is a commutative K-group (see [BLR], Def. 4.1/2).

More precisely, the K-group X_K should be considered as a K-torsor under X_K (in the sense of [BLR], chapter 6.4). Physically, this means that the special choice of a zero element of the group is forgotten as it should be for physical reasons. In order to generalize this notion slightly, one may also admit K-torsors X_K under K-groups $G_K \neq X_K$. However, we will restrict attention to the case $G_K = X_K$.

Definition 0.7 is motivated by *special relativity* with electromagnetism: The Minkowski space-time of special relativity naturally carries an additive, commutative group structure, and the gauge group of electromagnetism is commutative, too. This explains the label *model of type (SR)*, because (SR) shall remind of special relativity. One can prove that the following statements are true for *all* models of type (SR).

[1] Recall that the affine space $\mathbb{A}^n_S$ may be regarded as the algebraic geometric analogue of flat space. To see this, let $S = \operatorname{Spec} R$, i.e. $\mathbb{A}^n_S = \operatorname{Spec} R[T_1, \ldots, T_n]$ and consequently $\mathbb{A}^n_S(S) = \operatorname{Hom}_R(R[T_1, \ldots, T_n], R) \cong R^n$. In the special case $R = \mathbb{K}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the space-time induced by $\mathbb{A}^n_S$ is therefore the flat manifold $\mathbb{A}^n_S(S) = \mathbb{K}^n$.

0.8 Properties of models of type (SR). Let $(X \to S, g)$ be a model of type (SR). Then the following statements are true:

1. $X \to S$ is étale-invariant. More precisely, this statement means the following: Let $\varphi : X \to X$ be an étale S-morhpism, and let $(X' \to S'$, $g')$ be the pair obtained from $(X \to S, g)$ by base change with an étale morphism $S' \to S$. Then, $(X \to S, \varphi^* g)$ and $(X' \to S'$, $g')$ are models of type (SR), too.

   PROOF The crucial point is that Néron models are compatible with étale base change ([BLR], Prop. 1.2/2 c)).
2. X cannot be the flat, topologically trivial Minkowski space (see above).
3. The archimedean component $X_K(K)$ is bounded with respect to all p-adic norms. In the special case $K = \mathbb{Q}$ and under the assumption that there is a closed immersion $X_K \hookrightarrow \mathbb{A}^n_K$, this is the following statement: For each prime number p, the p-adic manifold $X_K(\mathbb{Q}_p)$ is a bounded subset of some $\mathbb{Q}_p^n$ with respect to the canonical p-adic norm $|\cdot|_p$.

   PROOF We know that X_K possesses a global Néron model. Consequently, the local Néron models exist ([BLR], Prop. 1.2/4). Then the boundedness assertion is due to [BLR], Thm. 10.2/1.
4. The archimedean component $X_K(K)$ carries a discrete geometry, if X_K is quasi-compact, because in this case X_K is an Abelian variety (due to the following statement e)). Therefore, the Mordell-Weil theorem tells us that $X_K(K)$ is a finitely generated abelian group, i.e.

   $$X_K(K) \cong \mathbb{Z}^d \oplus \mathbb{Z}/(p_1^{\nu_1}) \oplus \cdots \oplus \mathbb{Z}/(p_s^{\nu_s})$$

   for some prime numbers $p_i \in \mathbb{N}$ and integers $d, s, \nu_i \in \mathbb{N}$. In the special case $d = 0$, $X_K(K)$ consists of only finitely many points.
5. If we do not demand the quasi-compactness of the archimedean component X_K of X, one can prove that X_K possesses a Néron model if and only if there is an exact sequence

   $$0 \to T_K \to X_K \to A_K \to 0$$

   over some algebraic closure of K, where T_K is an algebraic torus and A_K is an abelian variety. While A_K is p-adically bounded, T_K is not. We interprete of A_K as space part and the torus T_K as an internal, gauge group part which we therefore associate with electromagnetism. Thus, X_K should appear as A_K-torsor under T_K (which is the algebraic geometric analogue of the differential geometric principal bundle of gauge theory).
6. On the "adelic" level there is some kind of entanglement of dimensions. For example, it is in general not possible to diagonalize the metric at the "adelic" points of $X(S)$.

Let us remark that the statements 1. and 2. are also true for models of type (GR).

3. Appendix: The arithmetic Einstein equations

Let $X \to S$ be a smooth, separated S-scheme. Consider the sheaf $\Omega^1_{X/S}$ of differential forms and the tangent bundle $T_{X/S} := \mathbb{V}(\Omega^1_{X/S})$. One can prove that sections of $T_{X/S}$ correspond to vector-fields: $\Gamma(T_{X/S}/X) := \mathrm{Hom}_X(X, T_{X/S}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X)$.

Definition. Let g be a global section of $\Omega^{\otimes 2}_{X/S}$. Locally, we may write $g|_U = \sum_{i,j} g_{ij}\, \omega^i \otimes \omega^j \in \Omega^{\otimes 2}_{X/S}(U)$, where $\{\omega^i\}$ is a local base of $\Omega^1_{X/S}(U)$, and where $g_{ij} \in \mathcal{O}_X(U)$. Then g is called a *metric* if $g_{ij} = g_{ji}$ and $\det(g_{ij}) \in \mathcal{O}_X(U)^*$ for all open $U \subset X$ such that $\Omega^1_{X/S}(U)$ is free.

Definition. Let $\nabla : T_{X/S} \times_X T_{X/S} \to T_{X/S}$ be an X-morphism. Then ∇ is called a *covariant derivation* if the following holds for all global sections $f \in \mathcal{O}_X(X)$ and $u, v \in \Gamma(T_{X/S}/X)$:

(i) $\nabla_{fu} v = f \nabla_u v, \qquad \nabla_{(u+u')} v = \nabla_u v + \nabla_{u'} v.$

(ii) $\nabla_u(fv) = (uf)v + f\nabla_u v, \qquad \nabla_u(v + v') = \nabla_u v + \nabla_u v'.$

In exactly the same way as in differential geometry, we may now straightforward introduce the notion of metrical connections ($ug(v,w) = g(\nabla_u v, w) + g(u, \nabla_u w)$), of torsion ($T(u,v) := \nabla_u v - \nabla_v u - [u,v]$) and of curvature ($R_{u,v}(w) := \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$). Thus we finally obtain the algebraic geometric analogue of the Levi-Civita connection and the Einstein equations. For more details, we refer the reader to arXiv:hep-th/0809.3579.

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