$\overline{SL}(5, R)$ fields in gravity and brane physics*

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Abstract

The $SA(5, R) = T_5 \wedge SL(5, R)$ group of the special affine transformations of the five dimensional space-time, and its universal (double) covering $\overline{SA}(5, R) = T_5 \wedge \overline{SL}(5, R)$ group, are considered in the fields of gravity and/or 4-branes. The homogenious $\overline{SL}(5, R)$ subgroup and its spinorial representations are essential for the construction of the relevant world, i.e. holonomic, spinorial fields.

The gauging of the $SA(5, R)$ group is presented, and the corresponding wave equations and the Bianci identities are presented in terms of the appropriate connection, torsion and curvature fields. These fields are given in terms of the $SO(1, 4)$ Lorentz-like subgroup and reduced w.r.t $SO(1, 3) \subset SO(1, 4)$ irreducible components, that determine the 4-dimensional particle content. The general $\overline{SL}(5, R)$ invariant lagrangian (linear and quadratic in terms of the gauge field strengths) coupled to the Higgs, spinorial and tensorial matter fields is considered. The spinorial matter fields are either holonomic, infinite-component ones defined by the spinorial representations of the $\overline{SL}(5, R)$ group, or anholonomic finite-component ones defined by the nonlinear representations of the $\overline{SL}(5, R)$ group under its $Spin(1, 4)$ subgroup. A spontaneous symmetry breaking mechanism, yielding the 4-dimensional Poincaré theory of gravity is designed.

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1. Introduction

A relativistic quantum field theory (RQFT) of gravitational interactions in D dimensions may involve the gauging of the $GA(D, R) = T_D \wedge GL(D, R)$

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group, to the extent that the answer can and should indeed be given within RQFT. Appropriate theories in $D = 4$ dimensions have been developed as Metric-Affine [1] and/or Gauge-Affine [2] ones, generalizing the, by now well established, Poincaré gauge field theory of gravity. They reduce, via spontaneous symmetry breaking mechanism, down to an Einstein-like General Relativity type of theory, thus being consistent with all basic gravity tests.

The basic wisdom of the standard approach to General Relativity is to start with the group of "general coordinate transformations" (GCT), i.e. the group of diffeomorphisms $Diff(4, R)$ of R^4. The theory is set upon the principle of general covariance. The GCT group has finite-dimensional tensorial representations only, and these representations characterize allowed world fields. A unified holonomic description of both tensors and spinors would require the existence of respectively tensorial and (double valued) spinorial representations of the GCT group. In other words one is interested in the corresponding single-valued representations of the double covering $\overline{GCT}$ of the GCT group, since the topology of GCT is given by the topology of its linear compact subgroup. It is well known that the finite-dimensional representations of $\overline{GCT}$ are characterized by the corresponding ones of the $\overline{SL}(4, R) \subset \overline{GL}(4, R)$ group, and $\overline{SL}(4, R)$ does not have *finite* spinorial representations. However, *there are infinite-dimensional* $\overline{SL}(4, R)$ *spinorial representations* that define the true "world" (holonomic) spinors [3].

There are two basic ways to introduce finite spinors in a generic curved space-time: i) One can make use of the nonlinear representations of the $\overline{GCT}$ group, which are linear when restricted to the Poincaré subgroup [4] with metric as a nonlinear realizer field. ii) One can introduce a bundle of cotangent frames, i.e. a set of 1-forms e^A (tetrads; $A = 0, 1, 2, 3$ the anholonomic indices) and define in this space an action of a physically distinct local Lorentz group. Owing to this Lorentz group one can introduce finite spinors, which behave as scalars w.r.t. $\overline{GCT}$. The bundle of cotangent frames represents an additional geometrical construction corresponding to the physical constraints of a local gauge group of the Yang-Mills type, in which the gauge group is the isotropy group of the space-time base manifold.

In order to set up a framework for a unified description of both tensors and spinors one is now naturally led to enlarge the local Lorentz group to the whole linear group $\overline{GL}(4, R)$, and together with translations one obtains the affine group $\overline{GA}(4, R)$. The affine group translates and deforms the tetrads of the locally Minkowskian space-time [5], and provides one with either infinite-dimensional linear or finite-dimensional nonlinear spinorial representations [6].

The subject of extended objects was initiated in the particle/field theory framework by the Dirac action for a closed relativistic membrane as the $(2 + 1)$-dimensional world-volume swept out in spacetime [7]. It evolved and become one of the central topics following the Nambu-Goto action for a closed relativistic string, as the $(1 + 1)$-dimensional worldsheet area swept out in spacetime [8]. An important step was the Polyakov action for a closed relativistic string, with auxiliary metric [9], that enabled consequent formulations of the Green-Schwarz superstring [10], and the bosonic, and super p-branes with manifest spacetime supersymmetry [11]. In this work, we follow the original path of the Nambu-Goto-like formulation of the bosonic p-brane and address the question of the spinors of the brane world-volume symmetries. For $p = 1$, these spinors are well known, and represent an important ingredient of the spinning string formulation and the Neveu-Schwarz-Ramond infinite algebras [8, 9].

There is a direct connection between the spinors appearing in the p-brane formulation and the world spinors of the Metric-Affine and Gauge-Affine theories of gravity, in a generic non-Riemannian spacetime of arbitrary torsion and curvature. This is due to a common geometric and group-theoretic structure of both theories.

In this work we concentrate on the Affine theories in $D = 5$, that are in its turn relevant for Kaluza-Klein-like generalizations of $D = 4$ Affine theories, as well as to certain features of a Dirac-like equation for $D = 4$ world spinors.

2. Affine gauge theory of gravity in $5D$

We gauge the general affine group $\overline{GA}(5,R) = T_5 \wedge \overline{GL}(5,R)$, a semidirect product of translations and the double-covering $\overline{GL}(5,R)$ of the general linear group $GL(5,R)$, generated by Q_{AB}. Here $\overline{GL}(5,R) = R_+ \otimes \overline{SL}(5,R) \supset R_+ \otimes \overline{SO}(1,4)$, where R_+ is the dilation subgroup. A general affine metric g_{AB} transforms as a 15 under global $GL(5,R)$: $\{g_{AB}\} = \{A^C_A A^D_B \eta_{CD} \mid A^B_A \in GL(5,R)\}$, where $A, B, \cdots = 0,1,2,3,4$ are the unholonomic (local) indeces. The Minkowski metric η is defined in a given "flat" $GL(5,R)$ gauge and is a Lorentz-like-subgroup invariant only. The decomposition of **15** w.r.t. the subgroup chain $GL(5,R) \supset SO(1,4) \supset SO(3,1) \supset SO(3)$ is given by: $15 = (\bar{1},\bar{1}) \oplus (\bar{0},\bar{0}) = (1,1) \oplus (\frac{1}{2},\frac{1}{2}) \oplus (0,0) \oplus (0,0) = 2 \oplus 1 \oplus 0 \oplus 1 \oplus 0 \oplus 0 \oplus 0$. Raising and lowering $\overline{GL}(5,R)$ indices with g_{AB} thus corresponds to the usual result when one goes to the flat gauge. The antisymmetric operators $Q_{[AB]} = \frac{1}{2}(Q_{AB} - Q_{BA})$ generate the Lorentz-like subgroup $\overline{SO}(1,4)$, the symmetric traceless operators (shears) $Q_{(AB)} = \frac{1}{2}(Q_{AB} + Q_{BA}) - \frac{1}{5} g_{AB} Q^C_C$ generate the proper 5-volume-preserving deformations while the trace $Q = Q^A{}_A$ generates scale-invariance R_+. $Q_{[AB]}$ and $Q_{(AB)}$ generate together the $\overline{SL}(5,R)$ group. The gauge potentials are the tetrads $e^A{}_M$ and connections $\Gamma^A{}_{BM}$, where $M, N, \cdots = 0,1,2,3,4$ are holonomic (curved space) indeces. The antisymmetric $\Gamma_{[AB]M}$, traceless symmetric $\Gamma_{(AB)M}$ and the trace $\Gamma^A{}_{AM}$, parts correspond, respectively, to the local Lorentz-like, shear and dilation transformations. The corresponding field strengths are the torsion

$$R^A{}_{MN} = \partial_M e^A{}_N + \Gamma^A{}_{BM} e^B{}_N - (M \leftrightarrow N),$$

and generalized "curvature"

$$R^A{}_{BMN} = \partial_M \Gamma^A{}_{BM} + \Gamma^C{}_{BM}\Gamma^A{}_{CN} - (M \leftrightarrow N).$$

Let ϕ be a generic matter field with a global general-affine matter lagrangian $L_m(\phi, \partial\phi)$. The total gauge-invariant lagrangian reads

$$L = L_m(\phi, \partial\phi, e, \Gamma) + L_g(e, \partial e, \Gamma, \partial\Gamma),$$

where Γ enters L_m through the covariant derivative

$$D_M = \partial_M - i\Gamma_{AM}^{\ B} Q_B^{\ A}.$$

The anholonomic expression for the covariant derivative (split w.r.t. Lorenz-like and shear connections is as follows,

$$D_A = e_A^M(\partial_M - \frac{i}{2}\Gamma^{[CD]}{}_M Q_{[CD]} - \frac{i}{2}\Gamma^{\{CD\}}{}_M Q_{\{CD\}}).$$

Variations of the frame and connection fields are:

$$\begin{aligned} \delta e^A &= D\epsilon^A - ie^C\epsilon^D R_{CD}^{\ A} \\ \delta\Gamma^{[AB]} &= D\epsilon^{[AB]} - ie^C\epsilon^D R^{[AB]}{}_{CD} \\ \delta\Gamma^{\{AB\}} &= D\epsilon^{\{AB\}} - ie^C\epsilon^D R^{\{AB\}}{}_{CD}, \end{aligned}$$

while the variation of a generic matter field ϕ reads,

$$\delta\phi = i(\epsilon^A D_A + \epsilon^{[AB]} Q_{[AB]} + \epsilon^{\{AB\}} Q_{\{AB\}}).$$

Here, $Q_{[AB]}$ and $Q_{\{AB\}}$ are appropriate, either finite tensorial or infinite tensorial/spinorial, expressions for the $\overline{SA}(5, R)$ generators in the space of matter-field components.

Variation with respect to ϕ yields the matter-field equations

$$\delta L_m / \delta\phi = 0.$$

Variation with respect to $e^a{}_\mu$ and $\Gamma^a{}_{b\mu}$ implies two (gravity) gauge field equations with the corresponding momentum and hypermomentum (angular momentum and deformation) currents as sources

$$\begin{aligned} -\delta L_g / \delta e^A{}_M &= e\Theta^M_A = \delta L_m / \delta e^A{}_M, \\ -\delta L_g / \delta\Gamma^A{}_{BM} &= e\Upsilon^{BM}_A = \delta L_m / \delta\Gamma^A{}_{BM}, \end{aligned}$$

where $e = \det(e_M^A)$ and (quasi) conservation laws:

$$\begin{aligned} &D_M(e\Theta_A^{\ M}) = e\Theta_B^{\ M} R^B{}_{AM} + e\Upsilon^{DM}_C R^C{}_{DAM}, \\ &D_M(e\Upsilon^B{}_A{}^M) = ee^B{}_M \Theta^M_A. \end{aligned}$$

The two gravity equations can be rewritten in the form

$$D_N \pi_A{}^{MN} - \epsilon_A^M = e\Theta_A^M,$$
$$D_N \pi^A{}_B{}^{MN} - \epsilon^A{}_B{}^M = e\Gamma^A{}_B{}^M,$$

where

$$\pi_A^{MN} = \partial L_g / \partial \partial_N e_M^A = 2\partial L_g / \partial R^A{}_{NM},$$
$$\pi^A{}_B{}^{MN} = \partial L_g / \partial \partial_N \Gamma^B{}_{AM} = 2\partial L_g / \partial R^B{}_{AMN},$$

and

$$\epsilon_A^M = e_A{}^M L_g - R^B{}_{AN} \pi_B{}^{NM} - R^B{}_{CAN} \pi^C{}_B{}^{NM},$$
$$\epsilon^A{}_B{}^M = e^A{}_N \pi_B{}^{NM}.$$

3. World and Affine matter fields in 5D

As the $\overline{SA}(5,R)$ subgroup of the $\overline{GA}(5,R)$ group determines the non-Abelian features, we restrict our attention to the former one. The $\overline{SA}(5,R)$ unitary irreducible representations (unirreps) are induced from the corresponding little group unirreps. The little group turns out to be $\overline{SA}(4,R)' = T_4' \wedge \overline{SL}(4,R)$, and thus we have the following possibilities:

(i) The whole little group is represented trivially, and we have a scalar state, which corresponds to a scalar field $\varphi(x)$, invariant under $\overline{SL}(5,R)$.

(ii) T_4' is represented trivially, and the corresponding states are described by the $\overline{SL}(4,R)$ unirreps, which are infinite-dimensional owing to the $\overline{SL}(4,R)$ noncompactness. The corresponding $\overline{SL}(5,R)$ states are therefore necessarily infinite-dimensional and when reduced with respect to the $\overline{SL}(4,R)$ subgroup should transform with respect to its unirreps.

(iii) The little group $\overline{SA}(4,R)'$ is represented nontrivially, and its representations are characterized by the $\overline{SA}(3,R)'' = T_3'' \wedge \overline{SL}(3,R)$ little group. When T_3'' is represented trivially we have states characterized by the $\overline{SL}(3,R)$ unirreps.

(iv) Finally, the little group $\overline{SA}(3,R)''$ is represented nontrivially and its representations are characterized by the $\overline{SA}(2,R)''' = T_2''' \wedge \overline{SL}(2,R)$ little group. When T_2''' is represented trivially we have states characterized by the $\overline{SL}(2,R)$ unirreps.

The finite-dimensional world tensor field components are characterized by the non-unitary representations of the homogeneous group $GL(5,R) \subset Diff(5,R)$. In the flat-space limit they split up into non-unitary $SO(1,4)$

irreducible pieces. The particle states are defined in the tangent flat-space only. They are characterized by the unitary irreducible representations of the (inhomogeneous) Poincaré-like group $P(5) = T_5 \wedge SO(1,4)$, and they are enumerated by the "little" group unitary representations (e.g. $T_4 \otimes SO(4)$ for $m \neq 0$). In the generalization to world spinors, the $SO(1,4)$ group is enlarged to the $\overline{SL}(5,R) \subset \overline{GL}(5,R)$ group, while $GA(5,R) = T_5 \wedge \overline{GL}(5,R)$ is to replace the Poincaré-like group. Affine "particles" are characterized by the unitary irreducible representations of the $\overline{GA}(5,R)$ group, whose unitarity is provided by the unitarity of the relevant "little" group (e.g. $T_4' \otimes \overline{SL}(4,R) \supset T_3'' \otimes SL(3,R) \supset T_2''' \otimes SL(2,R)$). A mutual particle–field matching is achieved by requiring the subgroup of the homogeneous group, that is isomorphic to the homogeneous part of the "little" group (say, $SO(4)$ of $SO(1,4)$), to be represented unitarily. Furthermore, one has to project away all representations of this group except a single one that is realized for the particle states (say $D^{(j_1,j_2)}$ of $SO(4) \subset T_4 \otimes SO(4)$).

A physically correct picture, in the affine case, is obtained by making use of the $\overline{SA}(5,R) \subset \overline{GA}(5,R)$ group unitary irreducible representations for "affine" particles, with particular states characterized by the $T_4 \otimes \overline{SL}(4,R)$ "little" group representations. The corresponding affine fields are described by the non-unitary infinite-dimensional $\overline{SL}(5,R) \subset \overline{GL}(5,R)$ representations, that are unitary when restricted to the homogeneous "little" subgroup $\overline{SL}(4,R)$. Therefore, the first step towards world spinor fields is a construction of infinite-dimensional non-unitary $\overline{SL}(5,R)$ representations, that are unitary when restricted to $\overline{SL}(4,R)$. These fields reduce to an infinite sum of (non-unitary) finite-dimensional $SO(1,4)$ fields.

The affine "particle" states transform according to the following representation

$$D((a,\bar{s})) \to e^{i(sp)\cdot a} D_{\{\overline{SL}\}}(L^{-1}(sp)\bar{s}L(p)), \quad (a,\bar{s}) \in T_4 \wedge \overline{SL}(5,R),$$

and $L \in \overline{SL}(5,R)/\overline{SL}(4,R)$ The unitarity properties of various representations in these expressions is as described above.

The $\overline{SL}(5,R)$ spinorial and/or tensorial fields transform as follows

$$(D(a,\bar{f})\Phi_A)(x) = (D_{\overline{SL}})^B_A(\bar{f})\Phi_B(f^{-1}(x-a)), \quad (a,\bar{f}) \in T_5 \wedge \overline{SL}(5,R),$$

Analogously, the world spinor fields transform w.r.t. $\overline{Diff}(5,R)$ as follows

$$(D(a,\bar{f})\Phi_{\tilde{A}})(x) = (D_{\overline{Diff_0}})^{\tilde{B}}_{\tilde{A}}(\bar{f})\Phi_{\tilde{B}}(f^{-1}(x-a)), \quad (a,\bar{f}) \in T_4 \wedge \overline{Diff_0}(5,R),$$

where $\overline{Diff_0}$ is the homogeneous part of $\overline{Diff}$, and $D_{\{\overline{Diff_0}\}} = \sum^{\oplus} D_{\{\overline{SL}\}}$.

4. Deunitarizing automorphism

Had the whole $\overline{SL}(5,R)$ been represented unitarily, the Lorentz-like boost generators would have a hermitian intrinsic part; as a result, when boosting a particle, one would obtain another particle - contrary to experience.

There exists however a remarkable inner "deunitarizing automorphism $\mathcal{A}$ [12], which leaves the $R_+ \otimes \overline{SL}(4,R)$ subgroup intact, and which maps the $Q_{(0k)}$, $Q_{[0,k]}$ generators into $iQ_{[0k]}$, $iQ_{(0k)}$ respectively ($k = 1,2,3,4$). The deunitarizing automorphism allows us to start with the unitary representations of the $\overline{SL}(5,R)$ group, and upon its application, to identify the finite (unitary) representations of the abstract $\overline{SO}(5)$ compact subgroup with nonunitary representations of the physical Lorentz-like group $\overline{SO}(1,4)$, while the infinite (unitary) representations of the abstract $\overline{SO}(1,4)$ group now represent (non-unitarily) the compact $\overline{SO}(5)/\overline{SO}(4)$ generators. The non-hermiticity of the intrinsic boost operators cancels their "intrinsic" physical action precisely as in finite tensors or spinors, the boosts thus acting kinetically only. In this way, we avoid a disease common to infinite-component wave equations.

5. Affine gravity lagrangian in $5D$

The first step towards construction of various possible gauge fields lagrangian expressions is to decompose torsion and curvature into irreducible pieces.

The torsion and curvature tensors decomposition into $\overline{SL}(5,R)$ irreducible pieces, as given by representation dimensionality, is as follows,

$$T^A{}_{MN} \to 35 \oplus 40 \oplus 40 \oplus 10'$$

$$R^{[AB]}{}_{MN} \to 105 \oplus 50 \oplus 45 \oplus 45 \oplus 5'$$

$$R^{\{AB\}}{}_{MN} \to 70 \oplus 105 \oplus 105 \oplus 50 \oplus 45$$

The decomposition of these $\overline{SL}(5,R)$ irreducible representations into the $\overline{SO}(5)$ irreducible pieces is as follows,

$$5' \to (\overline{\tfrac{1}{2}}, \overline{\tfrac{1}{2}})$$
$$10' \to (\overline{1}, \overline{0})$$
$$35 \to (\overline{\tfrac{3}{2}}, \overline{\tfrac{3}{2}}) \oplus (\overline{\tfrac{1}{2}}, \overline{\tfrac{1}{2}})$$
$$40 \to (\overline{\tfrac{3}{2}}, \overline{\tfrac{1}{2}}) \oplus (\overline{\tfrac{1}{2}}, \overline{\tfrac{1}{2}})$$
$$45 \to (\overline{\tfrac{3}{2}}, \overline{\tfrac{1}{2}}) \oplus (\overline{1}, \overline{0})$$
$$50 \to (\overline{2}, \overline{0}) \oplus 2 \times (\overline{1}, \overline{1}) \oplus 2 \times (\overline{0}, \overline{0}) \oplus (\overline{\tfrac{3}{2}}, \overline{\tfrac{1}{2}}) \oplus (\overline{\tfrac{1}{2}}, \overline{\tfrac{1}{2}})$$
$$70 \to (\overline{2}, \overline{2}) \oplus (\overline{1}, \overline{1}) \oplus (\overline{0}, \overline{0})$$
$$105 \to (\overline{2}, \overline{1}) \oplus (\overline{1}, \overline{1}) \oplus (\overline{1}, \overline{0})$$

It is important to note now that the pieces of dimensions 105, 70 and 35 contain $SO(3)$ components of spin greater than two, and thus we discard these irreducible pieces of torsion and curvature when considering possible lagrangian terms.

The gauge field lagrangian density consists besides the scalar curvature of quadratic terms of the following irreducible pieces written in terms of all upper indices,

$T^{\{A[B\}C}$, $T^{[ABC]}$, $R^{\{[AB][CD]\}}$, $R^{[C[A]\{B]D\}}$, $R^{[ABCD]}$, $R^{[\{AB\}\{CD\}]}$, $R^{\{A[B\}CD]}$

One can design spontaneous symmetry breaking scenarios to break the $\overline{SL}(5,R)$ symmetry either to the $\overline{SL}(4,R)$ subgroup and eventually then to the $\overline{S)}(1,3)$ Lorentz subgroup, or strait down to the Lorentz subgroup itself. The appropriate choices of the Higgs fields are:

$$\varphi \sim 5 \Rightarrow \overline{SL}(5,R) \to \overline{SL}(4,R)$$
$$\varphi \sim 15 \Rightarrow \overline{SL}(5,R) \to \overline{SO}(1,3)$$

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