Generalization of Gell-Mann formula for $sl(3,\mathbb{R})$ and $sl(4,\mathbb{R})$ algebra*

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Abstract

The so called Gell-Mann formula is an useful prescription that, in some cases, allows us to express algebra generators in terms of generators of the corresponding Inönü-Wigner contracted algebra. We demonstrate that in the cases of the $sl(3,\mathbb{R})$ and $sl(4,\mathbb{R})$ algebras, the Gell-Mann formula applies only for some representations. We point out how to generalize the Gell-Mann formula for the $sl(3,\mathbb{R})$ and $sl(4,\mathbb{R})$ algebras to be valid for all representations.

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1. Introduction

The Gell-Mann, or "decontraction" formula is a simple prescription that, in many cases, determines deformation of algebra that is "inverse" to the Inönü-Wigner contraction [1]. Namely, it gives a formula for generators of "decontracted" algebra in terms of generators of the contracted one. Since, by rule, properties of the contracted algebra (e.i. representations) are much easier to explore, this formula has a great potential value. As an useful and simple tool, this formula can be found even in some textbooks [2] and mathematical encyclopedia [3].

The main drawback of this formula is its limited validity. There is a number of references dealing with the question when this formula is applicable [2], [4], [5]. The formula was best studied in the case of contraction of (pseudo) orthogonal algebras: $so(M,N) \to R^{M+N-1} \wedge so(M,N-1)$ or $\to R^{M+N-1} \wedge$

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$so(M-1,N)$, where, loosely speaking, the Gell-Mann formula works very well [6]. At the same time, the case of (pseudo) orthogonal algebras is the only one where the formula is valid for (almost) all representations [7].

In this paper we are interested in Gell-Mann's formula in the case of $sl(N,\mathbb{R})$, where the contraction is done with respect to its maximal compact subalgebra $so(N)$. This case has an important application in finding representations of $sl(N,\mathbb{R})$ in physically interesting basis of $SO(N)$. Moreover, in this way it is possible to find directly all matrix elements of the $SU(N)$ representation in the $SO(N)$ basis. We will discuss particular cases of $sl(3,\mathbb{R})$ and $sl(4,\mathbb{R})$.

2. Inönü-Wigner contraction of $sl(N,R)$ algebra

Generators of the $sl(N,\mathbb{R})$ algebra can be split into two subsets: generators M_{ab} of the maximal compact subalgebra $so(N)$ (corresponding to antisymmetric real $N\times N$ matrices) and the, so called, sheer generators T_{ab} (corresponding to symmetric traceless real $N\times N$ matrices). Generators satisfy:

$$[M_{ab},M_{cd}] = i(\delta_{ac}M_{bd}+\delta_{ad}M_{cb}-\delta_{bc}M_{ad}-\delta_{bd}M_{ca}) \tag{1}$$

$$[M_{ab},T_{cd}] = i(\delta_{ac}T_{bd}+\delta_{ad}T_{cb}-\delta_{bc}T_{ad}-\delta_{bd}T_{ca}) \tag{2}$$

$$[T_{ab},T_{cd}] = i(\delta_{ac}M_{db}+\delta_{ad}M_{cb}+\delta_{bc}M_{da}+\delta_{bd}M_{ca}) \tag{3}$$

$$M_{ab}=-M_{ba}, T_{ab}=T_{ba}$$

Inönü-Wigner contraction of $sl(N,\mathbb{R})$ with respect to the maximal compact subalgebra $so(N)$ is given by the limiting procedure:

$$U_{ab} \equiv \lim_{a\to 0}(aT_{ab}), \tag{4}$$

which leads to the following algebra relations:

$$[M_{ab},M_{cd}] = i(\delta_{ac}M_{bd}+\delta_{ad}M_{cb}-\delta_{bc}M_{ad}-\delta_{bd}M_{ca}) \tag{5}$$

$$[M_{ab},U_{cd}] = i(\delta_{ac}U_{bd}+\delta_{ad}U_{cb}-\delta_{bc}U_{ad}-\delta_{bd}U_{ca}) \tag{6}$$

$$[U_{ab},U_{cd}] = 0. \tag{7}$$

Therefore, contraction of $sl(N,\mathbb{R})$ gives a $R_{\frac{N(N+1)}{2}-1}\wedge so(N)$ algebra, where $R_{\frac{N(N+1)}{2}-1}$ is an Abelian subalgebra (ideal) of translations in $\frac{N(N+1)}{2}-1$ dimensions.

The Gell-Mann formula, which is a prescription to make a "inverse" to (4), in this case reads:

$$T_{ab} = \sigma U_{ab} + \frac{i\alpha}{\sqrt{U\cdot U}}[C_2(SO(N)),U_{ab}], \tag{8}$$

where $C_2(SO(N))$ denotes the second order Casimir operator of the $SO(N)$ group ($\frac{1}{2}\sum M_{ab}M_{ab}$), σ is an arbitrary (complex) parameter and α is real normalization constant that depends on N.

In order to apply the Gell-Mann formula to obtain $sl(N,\mathbb{R})$ representations, we first need to find representations of the contracted algebra. Representations of the $so(N)$ generators M_{ab} are well known. There are two properties that determine representations of U_{ab}. First, generators U_{ab} transform according to $\frac{N(N+1)}{2}-1$ dimensional representation of $so(N)$ (6), i.e. as symmetric second order $so(N)$ tensors (in Young diagram notation $\square\square$). Second, generators U_{ab} mutually commute.

These two requirements, combined, imply that operator U_{ab} should be represented as an $SO(N)$ matrix element, seen as a function of group parameters:

$$U_{ab} \sim D^{(\square\square)}_{(cd)(ab)}(\theta) \equiv \left\langle \begin{array}{c}\square\square\\ (cd)\end{array} \middle| R(\theta) \middle| \begin{array}{c}\square\square\\ (ab)\end{array} \right\rangle . \tag{9}$$

$R(\theta)$ here denotes rotation given by set of $SO(N)$ parameters θ, while (ab) and (cd) are pairs of $SO(N)$ indices that, loosely speaking, specify one out of $\frac{N(N+1)}{2}-1$ vectors of $\square\square$ representation.[1] Representation space is space of square integrable functions over $SO(N)$ group, with standard invariant Haar measure: $\mathcal{L}^2(|g(\theta)\rangle)$. Generators U_{ab} act as multiplicative operators in this space, whereas action of $so(5)$ subalgebra generators is connected with the right group action in the standard way:

$$M_{ab}\,|\phi\rangle = -i\frac{d}{dt}\exp(itM_{ab})\Big|_{t=0}|\phi\rangle\,, \qquad g'\,|g\rangle = |g'g\rangle\,.$$

(This representation space is highly reducible, but in this paper we leave aside the question of reducibility of representations.)

It is obvious that functions (9) possess both properties required for U_{ab} representation - being ordinary functions of parameters θ they mutually commute, and they transform correctly under action of $SO(N)$ subgroup that is taken to act on righthand vector in (9) (with indices ab). Vector $\left| \begin{array}{c}\square\square\\ (ab)\end{array} \right\rangle$ on the right side of (9) is, thus, specified by the required transformation properties of U_{ab}. On the other hand, vector that stands to the left of the rotation operator can be an arbitrary vector from the representation $\square\square$. This means that for any choice of (cd) pair we obtain one representation of U_{ab} generators, and, hence, one representation of the whole contracted algebra. However, most of these representations belong to the same equivalence class (two vectors which can be connected by a rotation lead to equivalent representations).

[1]If we would insist on mathematical strictness, we should, at this point, introduce some formal basis of $SO(N)$, instead of using indices (ab). On the other hand, that would unnecessarily complicate notation.

A natural orthonormal basis in this space is given by properly normalized functions of $SO(N)$ matrix elements:

$$\left\{ \left| \begin{array}{c} J \\ k\,m \end{array} \right\rangle \equiv \int \sqrt{dim(J)} D^J_{km}(\theta) d\theta \, |g(\theta)\rangle \right\}, \tag{10}$$

where $d\theta$ is (normalized) invariant Haar measure and D^J_{km} is matrix element function

$$D^J_{km}(\theta) \equiv \left\langle \begin{array}{c} J \\ k \end{array} \right| R(\theta) \left| \begin{array}{c} J \\ m \end{array} \right\rangle.$$

J stands for set of labels of $SO(N)$ (more generally $Spin(N)$) irreducible representation, where k and m label particular vector within that representation. Normalization is chosen so that:

$$\left\langle \begin{array}{c} J \\ k\,m \end{array} \right| \left. \begin{array}{c} J' \\ k'\,m' \end{array} \right\rangle = \delta_{JJ'} \delta_{kk'} \delta_{mm'}. \tag{11}$$

The $so(N)$ generators action in this basis is given by the standard rotation matrices of the $SO(N)$ group for representation J, and, in general, can be written using the Clebsch-Gordan coefficients as:

$$\langle M_{ab} \rangle = \left\langle \begin{array}{c} J' \\ k'\,m' \end{array} \right| M_{ab} \left| \begin{array}{c} J \\ k\,m \end{array} \right\rangle = \delta_{JJ'} \sqrt{C_2(J)}\; C^{J\,[\square,\square]\,J'}_{m\;(ab)\;m'} \;. \tag{12}$$

In this basis it is also easy to calculate matrix elements of operators U_{ab}:

$$\begin{aligned} \langle U_{ab} \rangle &= \left\langle \begin{array}{c} J' \\ k'\,m' \end{array} \right| D^{\square\square}_{(cd)(ab)} \left| \begin{array}{c} J \\ k\,m \end{array} \right\rangle \\ &= \sqrt{dim(J')dim(J)} \int D^{J'*}_{k'm'}(\theta) D^{\square\square}_{(cd)(ab)}(\theta) D^J_{km}(\theta) d\theta \\ &= \sqrt{\tfrac{dim(J)}{dim(J')}} C^{\,J\;\square\square\;J'}_{\,k(cd)k'}\; C^{\,J\;\square\square\;J'}_{\,m(ab)m'} \end{aligned} \tag{13}$$

Matrix representation of the whole contracted algebra in this space is thus explicitly given by (12) and (13).

3. Application of original Gell-Mann formula to $sl(3,\mathbb{R})$ and $sl(4,\mathbb{R})$ algebras

Once we know an action of the contracted algebra, we may try to use the Gell-Mann formula in order to obtain the corresponding action of the $sl(N,\mathbb{R})$ generators in this space. We will consider here cases of the $sl(3,\mathbb{R})$ and $sl(4,\mathbb{R})$ algebras.

In the case of $SL(3,\mathbb{R})$, the maximal compact subgroup is $SO(3)$, and the representation space is given by finite norm linear combinations of basis vectors

$$\left\{ \left| \begin{array}{c} J \\ k\,m \end{array} \right\rangle, J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots ; k, m = -J, -J+1, \ldots, J \right\}. \tag{14}$$

The $SO(3)$ representation of the symmetric traceless tensor $\square\square$ corresponds to five dimensional representation $J = 2$. Any vector from this representation can be used to construct the U_{ab} operators. Calculations are most simple if we take $U_{ab} \sim D^{(\square\square)}_{(33)(ab)}$ or, in a more precise mathematical notation of the spherical $SO(3)$ basis, $U_\mu \sim D^{(\square\square)}_{0\mu}$.

By plugging this form of the contracted U operators in the Gell-Mann formula (8) we obtain

$$T_\mu = \sigma D^2_{0\mu} + \frac{i}{\sqrt{6}}[C_2(SO(3)), D^2_{0\mu}]. \tag{15}$$

This leads to the following expression for matrix elements of the shear generators:

$$\left\langle \begin{array}{c} J' \\ k'm' \end{array} \middle| T_\mu \middle| \begin{array}{c} J \\ k\,m \end{array} \right\rangle = \left(\sigma + i\frac{J'(J'+1)-J(J+1)}{\sqrt{6}}\right)\sqrt{\tfrac{2J+1}{2J'+1}}\, C^{J2J'}_{k0k'}\; C^{J\,2\,J'}_{m\mu m'} \tag{16}$$

The shear generators (15) satisfy relation (2) by construction, but relation (3) must be checked to hold. However, by direct calculation it is easy to find many violations of this relation. More precisely, it turns out that the $sl(3,\mathbb{R})$ algebra is correctly represented by (12) and (15), only if we constrain representation space to the subspace $k = 0$:

$$\left\{ \left| \begin{array}{c} J \\ 0\,m \end{array} \right\rangle, J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots; m = -J, -J+1, \ldots, J \right\}, \tag{17}$$

that is, if we restrict to $SL(3,\mathbb{R})$ representations over the $SO(3)/SO(2)$ coset space.

Representations defined over this subspace are the multiplicity free ones with respect to the compact subgroup $SO(3)$, since the number k, which effectively counts multiplicity (i.e. degeneracy of J values in $SL(3,\mathbb{R})$ representations) is fixed. The Gell-Mann prescription (15) does not work in the general case: comparison with known classification of the $SL(3,\mathbb{R})$ representations [8] reveals that the $SL(3,\mathbb{R})$ representations with multiplicity as well as the spinorial multiplicity free representations can not be obtained in this way ($k = 0$ implies that J must take strictly integer values).

In the case of $SL(4,\mathbb{R})$ the maximal compact subgroup is $SO(4)$. One way to choose a basis for $SO(4)$ representations is with respect to the decomposition $so(4) = so(3) \oplus so(3)$, i.e. $\left| \begin{array}{c} J_1\ J_2 \\ m_1 m_2 \end{array} \right\rangle$, where pairs (J_1, m_1) and (J_2, m_2) specify vectors of the two $SO(3)$ representations. A basis of the representation space (10) is then given by vectors

$$\left\{ \left| \begin{array}{c} J_1 J_2 \\ k_1 k_2 m_1 m_2 \end{array} \right\rangle, J_i = 0, \frac{1}{2}, \ldots; k_i, m_i = -J_i, -J_i+1, \ldots, J_i \right\}. \tag{18}$$

Similarly as in $SO(3)$ case, Gell-Mann formula can be applied to yield:

$$T_{\mu_1\mu_2} = \sigma D^{11}_{00\mu_1\mu_2} + \frac{i}{2}[C_2(SO(4)), D^{11}_{00\mu_1\mu_2}]. \tag{19}$$

This leads to matrix elements of the form:

$$\begin{gathered}\left\langle \begin{matrix} J'_1 J'_2 \\ k'_1 k'_2 m'_1 m'_2 \end{matrix} \middle| T_{\mu_1\mu_2} \middle| \begin{matrix} J_1 J_2 \\ k_1 k_2 m_1 m_2 \end{matrix} \right\rangle = \\ (\sigma + i\tfrac{1}{2} J'_1(J'_1+1) + J'_2(J'_2+1) - J_1(J_1+1) - J_2(J_2+1))\sqrt{\tfrac{(2J_1+1)(2J_2+1)}{(2J'_1+1)(2J'_2+1)}} \times \\ C^{J_1 1 J'_1}_{k_1 0 k'_1}\, C^{J_2 1 J'_2}_{k_2 0 k'_2}\, C^{J_1\ 1\ J'_1}_{m_1 \mu_1 m'_1}\, C^{J_2\ 1\ J'_2}_{m_2 \mu_2 m'_2}\ .\end{gathered} \tag{20}$$

However, much like as in the $SO(3)$ case, this formula is valid only in the subspace $k_1 = k_2 = 0$ (with J_1, J_2 integers) of (18), corresponding to the multiplicity free $SL(4)$ representations in spaces over the $(SU(2)/U(1)) \times (SU(2)/U(1))$ coset space.

Some other multiplicity free representations can additionally be obtained if we start from $SO(4)$ basis corresponding to decomposition $SO(4) \supset SO(3) \supset SO(2)$:

$$\left\{ \left| \begin{matrix} J_1 J_2 \\ J \\ m \end{matrix} \right\rangle, J_i = 0, \frac{1}{2}, \dots; J = |J_1 - J_2|, \dots J_1 + J_2; m = -J, \dots J \right\}. \tag{21}$$

Basis of $SL(4)$ representation space is then given by vectors:

$$\left\{ \left| \begin{matrix} J_1 J_2 \\ K\ \ J \\ k\ \ m \end{matrix} \right\rangle, J_i = 0, \frac{1}{2}, \dots; K, J = |J_1 - J_2|, \dots J_1 + J_2; k, m = -J, \dots J \right\}. \tag{22}$$

The corresponding Gell-Mann formula construction for shear generators,

$$T_{j,\mu} = \sigma D^{\ 11}_{\substack{0\, j \\ 0\, \mu}} + \tfrac{i\sqrt{3}}{4}[C_2(SO(4)), D^{\ 11}_{\substack{0\, j \\ 0\, \mu}}] \tag{23}$$

applies then only in the subspace of (22) with $K = k = 0$. This corresponds to the multiplicity free $SL(4)$ representations defined on symmetric spaces over $SO(4)/SO(3)$ coset space. These representations have only "diagonal" $SO(4)$ content, since the requirement $K = 0$ implies $J_1 = J_2$.

Note that neither representations with multiplicity nor spinorial multiplicity free representations (see [9]) can be obtained by making use of (19) or (23).

In these examples we have seen that the Gell-Mann formula works only in some subspaces of the most general representation space, but does not work in the general case. This situation can be better understood if we consider the so called left action of $SO(N)$ compact subgroup. This action is generated by a set of $SO(N)$ operators K_{ab} that act on the left (i.e. "k") indices of the basis vectors (10), unlike generators M_{ab} which act on the right indices. The operators K and M mutually commute, however, $K^2 = M^2$ is satisfied. Whereas the $sl(N,\mathbb{R})$ operators M_{ab} are invariant under this left action, the shear operators T_{ab}, constructed by using the Gell-Mann prescription are not. Namely, the transformation properties of the shear generators are inherited from the contracted operators U_{ab} and are determined by the vector standing on the left side of the rotation matrix in (9). Therefore, commutator of two shear generators, constructed by using the Gell-Mann prescription can not be, in general, expected to be invariant with respect to the left $SO(N)$ subgroup action – which is necessary if it is to coincide with some combination of M operators. This is a way to understand why the shear generators (15), (19) and (23) do not satisfy the commutation relation (3) in a generic space over the $SO(N)$ group parameters.

We have seen, however, that it is possible to make a restriction to some subspace, where only the invariant part of the $[T,T]$ commutator survives that is proportional to the rotation generator in this subspace. For example, in the $sl(3,\mathbb{R})$ case the restriction was made to subspace where $k = 0$. This indicates that $[T,T]$ commutator contains some unwanted terms that are effectively proportional to the left action generator K_0, which vanish upon this space restriction. Similar thing happens with commutators of the shear generators (19) and (23).

4. Generalization of the Gell-Mann formula

The above analysis motivated a question whether it is possible to add to the Gell-Mann formula some terms proportional to the generators of the left group action, that cancel the unwanted terms and "fix" the formula.

Indeed, in the case of $sl(N,\mathbb{R})$, this seems to be possible. Moreover, this "generalization" of the Gell-Mann formula in the $sl(3,\mathbb{R})$ case can be read out from the known matrix elements of the $sl(3,\mathbb{R})$ representations with multiplicity [8]:

$$\tilde{T}_\mu = \sigma D^2_{0\mu} + \frac{1}{\sqrt{6}}[C_2(Spin(3)), D^2_{0\mu}] + (D^2_{2\mu} - D^2_{-2\mu})K_0 + \delta(D^2_{2\mu} + D^2_{-2\mu}). \tag{24}$$

Here, by $\tilde{T}_\mu$ we have denoted expression for the "generalized Gell-Mann formula" with two terms added on the right side to the original Gell-Mann expression. It can be shown (by a direct calculation) that this expression for shear generators satisfies the commutation relation (3) in the whole vector space over $Spin(3)$ group parameters.

Apart from the generators of the left group action that now explicitly appear in the formula (24), there is an additional arbitrary complex parameter δ. Since these free parameters σ and δ are in direct connection with the labels of $sl(3,\mathbb{R})$ representations [2], existence of two arbitrary parameters reflects the fact that rank of $sl(3,\mathbb{R})$ is 2. Already by this argument, it is obvious that original Gell-Mann formula (where only one free parameter appears) can not account for all representations.

Appearance of terms in (24) with nonzero value of k (third projection of left action "angular momentum"), shows that, in general, action of generalized shear generator will change (mix) k value of states it acts upon. This means that, within the same representation, we can expect vectors with same J and different k numbers, that is, in general, representations with multiplicity will appear. Nevertheless, generalized expression (24) contains the original formula (15) as a special case: upon acting on $k = 0$ states, additional terms in (24) will vanish if we choose $\delta = 0$ and we will arrive at those multiplicity free representations that are obtained by using the expression (15). Additionally, some multiplicity free representations can be obtained by using the generalized formula (24) that can not be obtained from (15). For example, if we choose $\sigma = \frac{3}{2}; \delta = -\frac{1}{2}$ in (24), the following multiplicity free space, "build" as linear combinations of basis vectors (10) with different k values, turns out to be closed under the action of $\tilde{T}$:

$$
\begin{gathered}
span\Big\{ \left| \begin{smallmatrix} \frac{1}{2} \\ m \end{smallmatrix} \right\rangle' = \left| \begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2}\ m \end{smallmatrix} \right\rangle + \left| \begin{smallmatrix} \frac{1}{2} \\ -\frac{1}{2}\ m \end{smallmatrix} \right\rangle, \\
\left| \begin{smallmatrix} \frac{5}{2} \\ m \end{smallmatrix} \right\rangle' = \left| \begin{smallmatrix} \frac{5}{2} \\ \frac{5}{2}\ m \end{smallmatrix} \right\rangle + \sqrt{\tfrac{5}{2}} \left| \begin{smallmatrix} \frac{5}{2} \\ \frac{1}{2}\ m \end{smallmatrix} \right\rangle + \sqrt{\tfrac{5}{2}} \left| \begin{smallmatrix} \frac{5}{2} \\ -\frac{1}{2}\ m \end{smallmatrix} \right\rangle + \left| \begin{smallmatrix} \frac{5}{2} \\ -\frac{5}{2}\ m \end{smallmatrix} \right\rangle, \\
\left| \begin{smallmatrix} \frac{9}{2} \\ m \end{smallmatrix} \right\rangle' = \left| \begin{smallmatrix} \frac{9}{2} \\ \frac{9}{2}\ m \end{smallmatrix} \right\rangle + \left| \begin{smallmatrix} \frac{9}{2} \\ \frac{5}{2}\ m \end{smallmatrix} \right\rangle + \sqrt{\tfrac{14}{4}} \left| \begin{smallmatrix} \frac{9}{2} \\ \frac{1}{2}\ m \end{smallmatrix} \right\rangle + \sqrt{\tfrac{14}{4}} \left| \begin{smallmatrix} \frac{9}{2} \\ -\frac{1}{2}\ m \end{smallmatrix} \right\rangle + \left| \begin{smallmatrix} \frac{9}{2} \\ -\frac{5}{2}\ m \end{smallmatrix} \right\rangle + \left| \begin{smallmatrix} \frac{9}{2} \\ -\frac{9}{2}\ m \end{smallmatrix} \right\rangle, \ldots \Big\}.
\end{gathered}
$$

This is actually the space of the unitary irreducible spinorial (J content are half-odd integers) $sl(3,\mathbb{R})$ representation, and original the Gell-Mann formula does not apply in this space.

Similarly, the Gell-Mann formula can be generalized in the $sl(4,\mathbb{R})$ case. Again, by reading from the known matrix elements of the $sl(4,\mathbb{R})$ representations with multiplicity [9], we conclude:

$$
\begin{gathered}
T_{\mu_1\mu_2} = \sigma D^{11}_{00\mu_1\mu_2} + \frac{i}{2}[C_2(SO(4)), D^{11}_{00\mu_1\mu_2}] + \\
\delta_1(D^{11}_{11\mu_1\mu_2} + D^{11}_{-1-1\mu_1\mu_2}) + (D^{11}_{11\mu_1\mu_2} - D^{11}_{-1-1\mu_1\mu_2})(K^{10}_{00} + K^{01}_{00}) + \\
\delta_2(D^{11}_{-11\mu_1\mu_2} + D^{11}_{1-1\mu_1\mu_2}) + (D^{11}_{-11\mu_1\mu_2} - D^{11}_{1-1\mu_1\mu_2})(K^{10}_{00} - K^{01}_{00}). \quad (25)
\end{gathered}
$$

As the rank of $sl(4,\mathbb{R})$ algebra is 3, that is also the number of arbitrary complex parameters, $\sigma, \delta_1, \delta_2$ (only three real are independent) appearing in this generalized expression.

As in the $sl(3,\mathbb{R})$ case, it is obvious that by adjusting the parameters and by working in the subspace defined by $k_1 = k_2 = 0$, the generalized formula effectively reduces to the Gell-Mann formula (19). Although it

[2] They do not label irreducible representations uniquely, but values of Casimir operators are functions of these complex parameters.

is less obvious, the generalized expression contains, as a special case, the formula (23) as well.

In parallel to the $sl(3,\mathbb{R})$ case, the generalized Gell-Mann formula works for all representations - with and without multiplicity.

5. Conclusion

It is possible to draw some general conclusions based on the considered examples of the $sl(3,\mathbb{R})$ and $sl(4,\mathbb{R})$ algebras.

First of all, the Gell-Mann formula applied to $sl(N,\mathbb{R})$ works only in some subspaces of the spaces of the most general representations. This is obvious already at the level of counting the number of free parameters: the Gell-Mann formula has only one arbitrary parameter, whereas for a generic irreducible representations of $sl(N,\mathbb{R})$ there are $N-1$ (continuous) parameters among labels.

Secondly, by adding terms of the form $D^{\Box\Box}_{\mu\nu}K_\lambda$ the Gell-Mann formula can be generalized (i.e. fixed) to satisfy the commutation relation (3) in the whole representation space (10).

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