On the duality principle for null vectors*

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Abstract

In the first part of this paper we give a short overview on duality principle for definite vectors and its relations with pointwise pseudo-Riemannian Osserman manifolds. In the second part, we analyze few important examples on which duality principle holds, and a natural extension of the duality principle for null vectors is given.

1. Introduction

Let $(M, \langle\cdot\,,\,\cdot\rangle)$ be a $m-$dimensional pseudo-Riemannian manifold of the signature $(r, m-r)$. Let $\varepsilon_X = \langle X, X\rangle$ be the norm of the vector $X \in T_pM$. We say that a tangent vector X is spacelike, timelike, null, definite or unit if $\varepsilon_X > 0$, $\varepsilon_X < 0$, $\varepsilon_X = 0, X \neq 0$, $\varepsilon_X \neq 0$ or $|\varepsilon_X| = 1$, respectively. Let $\mathcal{S}^+M$ and $\mathcal{S}^-M$ ($\mathcal{S}M = \mathcal{S}^+M \cup \mathcal{S}^-M$) be the unit sphere bundles of spacelike and timelike tangent vectors in TM, and let $\mathcal{N}(M)$ be the null cone of nonzero null vectors.

Let ∇ be the Levi-Civita connection and let R be the associated Riemannian curvature tensor; $R(X,Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. The Jacobi operator $\mathcal{K}_X : Y \longrightarrow R(Y,X)X$ is a symmetric endomorphism of the tangent bundle TM. For non-null X, $\mathcal{K}_X$ preserves the orthogonal space $\{X\}^{\perp}$, and we will use the notation $\mathcal{K}'_X$ for the restriction of $\mathcal{K}_X$ to this space.

We say that $(M, \langle\cdot\,,\,\cdot\rangle)$ is *spacelike*, *timelike* or *null pointwise Osserman* if the characteristic polynomial of $\mathcal{K}_X$ is constant on $X \in \mathcal{S}_p^-M$, $\mathcal{S}_p^+M$, or $\mathcal{N}_p(M)$,[1] respectively. It is very well known that the notions pointwise spacelike Osserman and pointwise timelike Osserman are equivalent. We speak about *globally spacelike, timelike or null* Osserman manifolds if the

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[1] $\mathcal{S}_p^-M$, $\mathcal{S}_p^+M$, and $\mathcal{N}_p(M)$, are pseudo-spheres and null cone of tangent space T_pM.

characteristic polynomial of $\mathcal{K}_X$ is constant on $\mathcal{S}^+M$, $\mathcal{S}^-M$, or $\mathcal{N}(M)$, respectively. This definition implies if M is null Osserman, then necessarily $\mathcal{K}_X$ is nilpotent for $X \in \mathcal{N}(M)$, and consequently $\mathcal{K}_X$ has only the eigenvalue 0. It is clear if any spacelike or timelike Osserman manifold is necessarily null Osserman. The converse can fail in general, see, for example [10] in the Lorentzian setting.

In non-Riemannian setting Jordan normal form plays a crucial role, since eigenvalue structure does not determine the Jordan normal form of a symmetric linear operator. One says that $(M, \langle\cdot,\cdot\rangle)$ is *spacelike*, *timelike*, or *null Jordan Osserman* if the Jordan normal form of $\mathcal{K}_X$ is constant on $X \in \mathcal{S}^+M$, $\mathcal{S}^-M$, or $\mathcal{N}(M)$, respectively. While the notions of globally spacelike Osserman or globally timelike Osserman are the same, the notions of globally spacelike Jordan Osserman and globally timelike Jordan Osserman are inequivalent. For more details about this topic see [9] and [12]. In [11] it is proven that for a neutral $(2,2)$ manifold the notions of: pointwise spacelike Osserman manifold, pointwise spacelike Jordan Osserman manifold, pointwise timelike Jordan Osserman manifold and null Osserman manifold, are equivalent.

In Riemannian case it was shown, by Chi and by Nikolayevsky, that globally Osserman manifolds of dimension $n \neq 16$ are two-point homogeneous spaces[2]; and consequently for $n \neq 16$, this gives an affirmative answer to *Osserman conjecture* (see, [17], [18], [7], [13], [14], [15]). For a generalization of Osserman conjecture to the pseudo-Riemannian case, see [5].

It is convenient to work in algebraic settings. Let R be an *algebraic curvature operator.* This is a tensor satisfying the curvature symmetries

$$R(X,Y) + R(Y,X) = 0, \tag{1}$$

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \tag{2}$$

$$\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle. \tag{3}$$

One says R is an *Osserman (Jordan-Osserman) algebraic curvature tensor* if the associated Jacobi operator has characteristic polynomial (Jordan-form) constant on the unit pseudospheres $\mathcal{S}_p^-M$ and $\mathcal{S}_p^+M$. Usually, one proves results on the algebraic level, and then obtains corresponding conclusions in the geometric context.

In the Riemannian settings, **duality principle** is the following property of an Osserman algebraic curvature tensor R:

$\lambda \in \mathbb{R}$ *satisfies the duality principle if for any unit vectors* X *and* Y *holds*

$$\mathcal{K}_X Y = \lambda Y \quad \textit{if and only if} \quad \mathcal{K}_Y X = \lambda X.$$

We say that R *satisfies the duality principle if and only if every eigenvalue of Jacobi operator* $\mathcal{K}_X$ *, satisfies the duality principle.*

[2] Two-point homogeneous spaces are $\mathbb{R}^n, \mathbb{R}\mathrm{P}^n, \mathbb{S}^n, \mathrm{H}^n, \mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{H}^n, \mathbb{H}\mathrm{P}^n$, $\mathbb{H}\mathrm{H}^n$, $\mathbb{C}ay\,\mathrm{P}^2$, and $\mathbb{C}ay\mathrm{H}^2$.

Duality principle is used as one of important tools in the proof of Osserman conjecture for $n \neq 8, 16$ by Nikolayevsky (see [14]).

Osserman manifolds are described in Riemannian setting (except some cases for $n = 16$), in the Lorentzian setting (spaces of constant sectional curvature, see [3]), but in the case of higher signature (except the case of 4-dimensional Kleinian manifolds, see [4], [9]) we are very far from the complete picture. For example, Osserman conjecture doesn't hold, see [19], [6], [4], [9]. The interesting questions which arise from our investigations are: classification of all algebraic curvature tensors which satisfy duality principle and classification of all pseudo-Riemannian manifolds whose curvature tensor satisfy duality principle.

Our paper is organized as follows. Section 1 is devoted to the introduction and motivation in this topic. Since k-stein manifolds are closely related to the Osserman manifolds, in Section 2 we give some basic characterizations of k-stein conditions and specially of 1-stein and 2-stein manifolds. Also, we introduce notion of diagonalizable pseudo-Riemannian Osserman manifolds and we obtain the analogous conditions (Theorem 2.) as in Riemannian case, see [20]. In Section 3 we recall of our main results (without proofs) given in [1]: the definition of duality principle for all non-null vectors (Theorem 4.), a characterization of duality principle for diagonalizable pseudo-Riemannian manifolds, and consequences of it. Section 4 contains original results and it is devoted to the examples of manifolds in which duality principle holds. After that, we investigate duality principle for null vectors, and find several examples of manifolds which allow an extension of duality principle to null vectors, also. Those examples, motivate us to extend the duality principle to null vectors.

2. Duality for non null vectors

2.1. k-stein

We say that a manifold M is *k-stein* if there exist constants C_t for $1 \leq t \leq k$ such that for all $X \in \mathcal{S}(M)$ hold $\mathrm{Tr}(\mathcal{K}_X^t) = (\varepsilon_X)^t C_t$. It is well-known that 1-stein manifolds are Einstein and vice versa and also that Osserman timelike (spacelike) condition is equivalent with k-stein condition (see for example [9, 1.7.3 Lemma]).

Let $(E_1, E_2, ..., E_m)$ be an arbitrary pseudo-orthonormal basis of T_pM. Let $X = \alpha E_i + \beta E_j$ where i and j are fixed and $1 \leq i \neq j \leq m$. Then we have

$$\mathcal{K}_X = \alpha^2 \mathcal{K}_i + \alpha\beta \mathcal{K}_{ij} + \beta^2 \mathcal{K}_j \tag{4}$$

where we put

$$\mathcal{K}_i = \mathcal{K}_{E_i} \quad \text{and} \quad \mathcal{K}_{ij} = R(\cdot, E_i)E_j + R(\cdot, E_j)E_i\,.$$

Now, after the substitution $\alpha^2 = \varepsilon_X\, \varepsilon_i - \beta^2 \varepsilon_i\, \varepsilon_j$ in (4) we obtain

$$\mathcal{K}_X = \varepsilon_X \varepsilon_i \mathcal{K}_i + \alpha\beta \mathcal{K}_{ij} + \beta^2(\mathcal{K}_j - \varepsilon_i \varepsilon_j \mathcal{K}_i). \tag{5}$$

Let us introduce the following notations,

$$A^i_{pq} := [\mathcal{K}_i]_{pq} = \varepsilon_p R_{qiip}\,, \qquad B^j_{pq} := [\mathcal{K}_j]_{pq} = \varepsilon_p R_{qjjp}\,,$$
$$Z^{ij}_{pq} := [\mathcal{K}_{ij}]_{pq} = \varepsilon_p (R_{qijp} + R_{qjip})\,. \tag{6}$$

Since

$$[\mathcal{K}^k_X]_{pq} = \sum_{p_2,p_3,\dots,p_k} [\mathcal{K}_X]_{pp_2}[\mathcal{K}_X]_{p_2p_3}\cdots[\mathcal{K}_X]_{p_kq}\,,$$
$$\mathrm{Tr}(\mathcal{K}^k_X) = \sum_{p_1,p_2,\dots,p_k} [\mathcal{K}_X]_{p_1p_2}[\mathcal{K}_X]_{p_2p_3}\cdots[\mathcal{K}_X]_{p_kp_1}\,,$$

the equation (5) put the k-stein condition in the following form

$$\varepsilon^k_X C_k = \sum_{p_1,\dots,p_k,p_{k+1}=p_1} \prod_{1\le t\le k} \Big(\varepsilon_X \varepsilon_i A^i_{p_tp_{t+1}} + \alpha\beta Z^{ij}_{p_tp_{t+1}} + \beta^2 (B^j_{p_tp_{t+1}} - \varepsilon_i\varepsilon_j A^i_{p_tp_{t+1}})\Big) \tag{7}$$

Lemma 1. (i1) *A manifold M is 1-stein iff for all $1 \le i \ne j \le m$, the following formulas hold*

$$\sum_{1\le p\le m} Z^{ij}_{pp} = 0 \qquad \textit{and} \qquad \sum_{1\le p\le m} (B^j_{pp} - \varepsilon_i\varepsilon_j A^i_{pp}) = 0. \tag{8}$$

(i2) *If a manifold is 2-stein then for all $1 \le i \ne j \le m$, the following formulas hold*

$$\sum_{1\le p,q\le m} A^i_{pq} Z^{ij}_{qp} = \sum_{1\le p,q\le m} B^j_{pq} Z^{ij}_{qp} = 0\,, \tag{9}$$
$$2 \sum_{1\le p,q\le m} A^i_{pq} B^j_{qp} - 2\,\varepsilon_i\varepsilon_j \sum_{1\le p,q\le m} A^i_{pq} A^i_{qp} + \sum_{1\le p,q\le m} Z^{ij}_{pq} Z^{ij}_{qp} = 0\,, \tag{10}$$
$$\sum_{1\le p,q\le m} A^i_{pq} A^i_{qp} = \sum_{1\le p,q\le m} B^j_{pq} B^j_{qp}\,. \tag{11}$$

Proof. The proof for both statements follows from (7) for $k = 1$ and $k = 2$, respectively. ♢

2.2. Diagonalizable manifolds

When the Jacobi operator $\mathcal{K}_X$ is diagonalizable for any $X \in \mathcal{S}M$, we call such manifolds *diagonalizable Osserman* (pseudo-Riemannian) manifolds. Since all information about R are encoding in the terms with $\alpha^i\beta^{i+2t}$ ($i = 0, 1$ and $t \in \mathbb{N}$) of $\mathrm{Tr}(\mathcal{K}^k_X)$ (see (7)), one can consider terms with $\alpha\beta$ and β^2. Following the proof of main theorem (Theorem 1.1) of [20], we proved the following theorem, in [1].

Theorem 2. *Let $(M, \langle \cdot\,, \cdot \rangle)$ be a diagonalizable pseudo-Riemannian Osserman manifold, and let $(E_1, E_2, ..., E_m)$ be an orthonormal basis of T_pM such that $\mathcal{K}_1$ has diagonal matrix with respect to this basis, and let $\Lambda_a = \{t \mid [\mathcal{K}_1]_{tt} = a\}$. Then for every eigenvalue a of $\mathcal{K}_1$, and for all $1 \le i \ne j \le m$, hold*

$$\text{(i1)} \qquad \sum_{t\in\Lambda_a} Z^{ij}_{tt} = 0, \tag{12}$$

$$\text{(i2)} \qquad \sum_{s,t\in\Lambda_a} Z^{ij}_{st}\, Z^{ij}_{st} = 0\,. \tag{13}$$

In [1] we generalize the duality principle (from [20]) in pseudo-Riemannian settings for non null vectors.

Definition 3. *Let R be an Osserman algebraic curvature tensor. For $\lambda \in \mathbb{R}$ we say that it satisfies the duality principle if for all mutually orthogonal unit vectors $X\,, Y$ holds*

$$\mathcal{K}_X(Y) = \varepsilon_X\, \lambda\, Y \quad \Longrightarrow \quad \mathcal{K}_Y(X) = \varepsilon_Y \lambda\, X\,. \tag{14}$$

If the duality principle holds for all $\lambda \in \mathbb{R}$ then we say that duality principle holds for the algebraic curvature tensor R (or for the pseudo-Riemannian Osserman manifold $(M, \langle \cdot\,, \cdot \rangle)$ whose curvature tensor is R).

In the following theorem we showed (see [1]) that we can relax condition on vectors X, and Y.

Theorem 4. *Let $(M, \langle \cdot\,, \cdot \rangle)$ be a diagonalizable Osserman manifold such that the duality principle holds for $\lambda \in \mathbb{R}$. Then implication (14) holds for all $X, Y \in T_pM$ with $\varepsilon_X \ne 0$.*

Let $(M, \langle \cdot\,, \cdot \rangle)$ be a diagonalizable pseudo-Riemannian Osserman manifold and let $(E_1, E_2, ..., E_m)$ be a pseudo-orthonormal basis of T_pM such that the Jacobi operator $\mathcal{K}_1$ has diagonal matrix with respect to this basis. Then the duality applied to the coordinate eigenvectors vectors gives

$$\mathcal{K}_1(E_j) = \varepsilon_1 \lambda\, E_j \Longrightarrow \mathcal{K}_j(E_1) = \varepsilon_j \lambda\, E_1. \tag{15}$$

$\mathcal{K}_1(E_j) = \varepsilon_1 \lambda\, E_j$ is equivalent to $\mu_j = A^1_{jj} = \varepsilon_1 \lambda$ and $A^1_{kj} = 0$ for $k \ne j$. From (6) it follows $\lambda = \varepsilon_1\, \varepsilon_j\, R_{j11j}$ and $R_{j11k} = 0$ for $k \ne j$. From the other hand we have $\mathcal{K}_j(E_1) = \varepsilon_j \lambda\, E_1$ is equivalent to $B^j_{11} = \varepsilon_j \lambda$ and $B^j_{k1} = 0$ for $k \ne 1$, then (6) gives $\lambda = \varepsilon_j\, \varepsilon_1\, R_{1jj1}$ and $R_{1jjk} = 0$ for $k \ne 1$. Since $\varepsilon_1 \varepsilon_j\, R_{j11j} = \varepsilon_j \varepsilon_1\, R_{1jj1}$ hold, we see that

$$(R_{j11k} = 0 \text{ for } k \ne j) \Longrightarrow (R_{1jjk} = 0 \text{ for } k \ne 1)\ , \tag{16}$$

is the sufficient condition for (15). Now, we can formulate the following theorem.

Theorem 5. *Let* $(M, \langle \cdot\,,\,\cdot \rangle)$ *be a diagonalizable pseudo-Riemannian Osserman manifold, and let* $(E_1, E_2, ..., E_n)$ *be an orthonormal basis of* T_pM *such that* $\mathcal{K}_1$ *has diagonal matrix with respect to this basis. The duality principle holds for* M *iff for all* $j > 1$ *and all* $1 \leq p \leq n$ *hold* $Z^{1j}_{pp} = 0$.

Now, we can combine the previous theorem and Theorem 4. to obtain some conditions on a diagonalizable pseudo-Riemannian manifolds under which duality principle holds.

Corollary 6. *The duality principle holds in a diagonalizable pseudo-Riemannian Osserman manifold* $(M, \langle \cdot\,,\,\cdot \rangle)$ *if*

(i1) its Jacobi operator has all different eigenvalues.

(i2) for every $X \in \mathcal{S}_pM$ *doesn't exist null eigenvector of* $\mathcal{K}_X$, *(specially in Riemannian case).*

The signature $(2, 2)$ gives the examples [3] of non-diagonalizable Osserman manifolds, but the duality holds, as the following theorem shows.

Theorem 7. *The duality principle holds for every* 4*-dimensional Osserman manifold.*

The general theory of symmetric endomorphisms in pseudo-unitarian spaces (see for example [16]), shows that for similar investigations of non-diagonalizable pseudo-Riemannian (Jordan) Osserman manifolds, we essentially need to study the duality principle for null vectors.

3. Duality for non-zero null vectors

Since duality principle holds in Riemannian, Lorentzian and four-dimensional manifolds (see [20], [3]), here we will investigate it on some examples in more general settings. Also, in all mentioned examples we will take care about behavior of null vectors.

3.1 Manifolds with constant sectional curvature

Let M be a pseudo-Riemannian manifold of signature $(r, m - r)$ which has constant sectional curvature λ. The curvature tensor of M is given by

$$R(X, Y)Z = \lambda\,(\langle Y, Z\rangle X - \langle X, Z\rangle Y).$$

If M is complete, connected and simply connected, $M = M(c, r, m - r)$ is determined by (c, r, q). These manifolds have been classified by Wolf [21]. Now, we want to examine duality for null vectors. It is known ([12, 1.7.4 Lemma]) that $\mathcal{K}_X$ is nilpotent for a null vector X, since M is m-stein. Then, the following theorem holds,

[3] of the smallest possible dimension.

Theorem 8. *Let M be a pseudo-Riemannian manifold of signature $(r, m-r)$ with constant sectional curvature $\lambda \neq 0$. Then duality principle holds for M. Moreover, for an arbitrary null vector X we have*

(i1) $\mathcal{K}_X^2 = 0$, *and* $\dim \mathsf{Ker}(\mathcal{K}_X) = m - 1$.

(i2) *if* $0 \neq Y$ *is a vector such that* $\mathcal{K}_X(Y) = 0$ *then* $\mathcal{K}_Y(X) = \lambda\, \varepsilon_Y X$.

Remark 2. Since the proof of this theorem is similar (but simpler), than the proof of Theorem 9., we will omit it.

4.2 Kähler manifolds with constant holomorphic sectional curvature.

The indefinite projective space $\mathbb{CP}_s^m(\lambda)$ of signature $(2s, 2m-2s)$ can be constructed, see [2] for details. The Kähler space form $\mathbb{CP}_s^m(4\,\lambda)$ has constant holomorphic sectional curvature $4\,\lambda \neq 0$ with curvature tensor:

$$\begin{aligned} R(X,Y)Z = \lambda\,\{&\langle Y, Z\rangle X - \langle X, Z\rangle Y \\ &+ \langle JY, Z\rangle\, JX - \langle JX, Z\rangle\, JY - 2\,\langle JX, Y\rangle\, JZ\}. \end{aligned} \tag{17}$$

Furthermore, every Kähler manifold M of signature $(2s, 2m-2s)$ with constant holomorphic sectional curvature $4\,\lambda \neq 0$ is holomorphically isometric to $\mathbb{CP}_s^m(4\,\lambda)$. The following theorem holds.

Theorem 9. *Let M be a Kähler manifold M of signature $(2s, 2m-2s)$ with constant holomorphic sectional curvature $4\,\lambda \neq 0$. Then duality principle holds for M. If X is an arbitrary null vector, then*

(i1) $\mathcal{K}_X^2 = 0$, *and* $\dim \mathsf{Ker}(\mathcal{K}_X) = 2\,m - 2$.

(i2) *if* $0 \neq Y$ *is a vector such that* $\mathcal{K}_X(Y) = 0$ *then* $\mathcal{K}_Y(X) = \lambda\, \varepsilon_Y X$.

Proof. Let J be a (almost) complex structure on M then for any vector X holds $\langle X, J(X)\rangle = 0$, and if X is null vector (17) is reduced to

$$\mathcal{K}_X(Y) = R(Y,X)X = \lambda\,\{-\langle Y, X\rangle X - 3\,\langle JY, X\rangle\, JX\}. \tag{18}$$

From (18) we have $X,\, J(X) \in \mathsf{Ker}\,(\mathcal{K}_X)$ and $\mathsf{Im}\,(\mathcal{K}_X) = \mathsf{span}\{X, J(X)\}$, which implies that $\mathcal{K}_X^2 = 0$. Since ± 1 is not eigenvalue of the operator J, the vectors X and $J(X)$ are linearly independent. It implies $\dim \mathsf{Ker}\,(\mathcal{K}_X) = 2\,m - 2$.

We can choose an orthonormal frame $(E_1, E_2, \ldots, E_{2m})$ of T_pM such that $\{E_1, \ldots, E_{2s}\}$ are timelike vectors, $\{E_{2s+1}, \ldots, E_{2m}\}$ are spacelike vectors and $JE_{2i-1} = E_{2i}$, $JE_{2i} = -\,E_{2i-1}$, for $i = 1, 2 \ldots m$. Then generic nonvanishing components of curvature tensor given by (17) are

$$\begin{aligned} &R_{2i-1\,2t-1\,2t\,2i} = -\,R_{2i-1\,2t\,2t-1\,2i} = \varepsilon_{2i}\,\varepsilon_{2t-1}\,\lambda, \quad t \neq i, \\ &R_{kjjk} = \begin{cases} 4\,\lambda, & \text{if } k = 2\,t,\ j = 2\,t-1 \ \text{ or } \ k = 2\,t-1,\ j = 2\,t, \\ \varepsilon_j\,\varepsilon_k\,\lambda, & \text{otherwise}, \end{cases} \end{aligned} \tag{19}$$

where $\varepsilon_i = \|E_i\|$. Then the matrix of Jacobi operator $\mathcal{K}_{E_i}$ with respect to this basis has the following diagonal form

$$\mathcal{K}_{E_i} = \begin{cases} \varepsilon_i \operatorname{diag}[\,\lambda, \lambda, \ldots, \overset{i}{0}, 4\,\lambda, \lambda, \ldots, \lambda\,]\,, & \text{if } i \text{ is odd}\,, \\ \varepsilon_i \operatorname{diag}[\,\lambda, \lambda, \ldots, 4\,\lambda, \overset{i}{0}, , \lambda, \ldots, \lambda\,]\,, & \text{if } i \text{ is even}\,. \end{cases} \tag{20}$$

Now, if we take an unit timelike vector X, then we choose a basis as above taking $X = E_1$. From (19) it follows that all components of the curvature tensor with three different indices (with respect to the chosen basis) are vanishing, and since the matrix of $\mathcal{K}_{E_i}$ is diagonal,(20), we conclude that the duality principle holds. The analogous proof works for any unit spacelike vector X.

For (i2), take any null vector $X \neq 0$ then $X = \alpha\,(E + F)$ where $\alpha \neq 0$ and E, F are unit timelike and spacelike vectors, respectively. The vectors X, and $J(X)$ are mutually orthonormal and linearly independent, and we can choose orthonormal basis like above where we take $E_1 = E$, $E_{2m} = F$, and then direct calculations shows that $\mathcal{K}_X$ in the basis $(E_1, E_2, E_{2m-1}, E_{2m}, E_3, \ldots, E_{2m-2})$ has matrix $\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ where

$$A = \begin{bmatrix} \lambda & 0 & 0 & -\lambda \\ 0 & -3\,\lambda & -3\,\lambda & 0 \\ 0 & 3\,\lambda & 3\,\lambda & 0 \\ \lambda & 0 & 0 & -\lambda \end{bmatrix}.$$

Then eigenvectors of $\mathcal{K}_X$ are $v_1 = X$, $v_2 = J(X) = E_2 - E_{2m-1}$, $v_j = E_j$ for $j = 3, \ldots, 2m-2$. Since, $\mathcal{K}_X^2 = 0$ and $\mathsf{rank}\,(\mathcal{K}_X) = 2$ we see that the Jordan normal form of $\mathcal{K}_X$ consists of two two-dimensional blocks $N_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and all other blocks are one-dimensional (zero matrices).

Let $Y = (y_1, y_2, \ldots, y_{2m})$ be an arbitrary eigenvector of $\mathcal{K}_X$, then it has form
$Y = (y_1, y_2, -y_2, y_1, y_5, \ldots, y_{2m})$. Let $z = J_Y(X)$, then using the components of the curvature tensor R given by (19) one can find that the only non-vanishing components of the vector z are $z_1 = z_4 = \lambda\,\langle Y, Y\rangle$. It implies (i2). $\diamondsuit$

3.3. Para-Kähler manifolds with constant para-holomorphic sectional curvature.

The tangent bundle $T\mathbb{S}^m$ of the standard sphere can be equipped with a pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$ of signature (m, m) and a para-complex structure such that $P^m(B) = (T\mathbb{S}^m, \langle\cdot,\cdot\rangle, F)$ is of constant para-holomorphic sectional curvature $4\,\lambda \neq 0$. For $m > 1$, $P^m(B)$ is complete, connected and simple connected, see [8] for details. Furthermore, every para-Käehler manifold M^{2m} with constant para-holomorphic sectional curvature $4\,\lambda$ is F holomorphically isometric to $P^m(B)$. The curvature tensor of $P^m(B)$

is given by

$$R(X,Y)Z = \lambda\{\langle Y,Z\rangle X - \langle X,Z\rangle Y \\ - \langle JY,Z\rangle\, JX + \langle JX,Z\rangle\, JY + 2\,\langle JX,Y\rangle\, JZ\}. \tag{21}$$

Again, we want to examine duality for null vectors, and we have,

Theorem 10. *Let M be a para-Kähler manifold of signature (m,m) with constant para-holomorphic sectional curvature $4\lambda \neq 0$. Then duality principle holds for M. If X is an arbitrary null vector, then*

(i1) $\mathcal{K}_X^2 = 0$.

(i2) $\dim \mathsf{Ker}(\mathcal{K}_X) = \begin{cases} 2m-2, & \text{if } J(X) \neq \pm X, \\ 2m-1, & \text{if } J(X) = \pm X. \end{cases}$

(i3) *if $0 \neq Y$ is a vector such that $\mathcal{K}_X(Y) = 0$ then $\mathcal{K}_Y(X) = \lambda\,\varepsilon_Y\,X$.*

The proof is similar to the proof of Theorem 9. and we will omit it. The only essential difference is a consequence of the fact that in the para-Kählerian manifolds there exist null eigenvectors of J for the eigenvalues ± 1. It means that for a such null vector X $(J(X) = \pm X)$, we have $\mathsf{Im}\,(\mathcal{K}_X) = \mathsf{span}\{X\}$, i.e., $\mathsf{rank}\,(\mathcal{K}_X) = 1$, and consequently the Jordan normal form of $\mathcal{K}_X$ consists of one two-dimensional block $N_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and all other matrix elements are zeros.

Let us remark that $\mathsf{rank}\,\mathcal{K}_X = 1$ for $X \in V_J(\pm 1) \cap \mathcal{N}_p(M)$, where $V_J(1)$, $V_J(-1)$ are eigenspaces of para-complex structure J for the eigenvalues 1 and -1 respectively, and $\mathcal{N}_p(M)$ is null cone of T_pM.

3.4. On duality for nonzero null vectors.

The above examples and results of previous sections (Theorem 4.), as well as the fact that for m-stein ($m = \dim M$) manifolds the Jacobi operator $\mathcal{K}_X$ of an arbitrary null vector $X \neq 0$ is nilpotent, motivate us to extend our definition of duality principle to null vectors. More precisely,

Definition 11. *For $\lambda \in \mathbb{R}$ we say that it satisfies the duality principle if for all vectors $X \neq 0 \neq Y \in T_p(M)$ such that*

(d) $\varepsilon(X) \neq 0$, *holds*

$$\mathcal{K}_X(Y) = \varepsilon_X \lambda Y \quad \Longrightarrow \quad \mathcal{K}_Y(X) = \varepsilon_Y \lambda\, X\,.$$

(n) $\varepsilon(X) = 0$, *holds*

$$\mathcal{K}_X(Y) = 0 \quad \Longrightarrow \quad X \textit{ is an eigenvector of } \mathcal{K}_Y\,,$$

and moreover if Y is null vector then

$$\mathcal{K}_X(Y) = 0 \quad \Longrightarrow \quad \mathcal{K}_Y(X) = 0\,.$$

If the duality principle holds for all $\lambda \in \mathbb{R}$ then we say that duality principle holds for the algebraic curvature tensor R (or for the pseudo-Riemannian Osserman manifold $(M, \langle \cdot\, , \cdot \rangle)$ whose curvature tensor is R).

Remark 3. Let us mention here that the first statements (i1) of Theorems 8., and 9., means that pseudo-Riemannian manifolds with constant scalar sectional curvatures and Kähler manifolds with constant holomorphic sectional curvatures are examples of globally null Jordan-Osserman manifolds. Also, they are manifolds in which duality principle holds for all type of vectors. Para-Kähler manifolds of dimension $2\,m$ with constant para-holomorphic sectional curvatures are manifolds in which duality principle holds for all type of vectors, but they are not null pointwise Jordan-Osserman manifolds, since their Jacobi operator change its Jordan form on the intersection of null cone and eigenspaces of para-complex structure J.

Remark 4. Let us consider the following Walker metric on $\mathbb{R}^4$ which are given in [9], pages 64-66,

$$g = \begin{pmatrix} x_3 f_1 & a & 1 & 0 \\ a & x_4 f_2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{22}$$

where $f_1 = f_1(x_1, x_2)$ and $f_2 = f_2(x_1, x_2)$ are smooth real functions and where a is constant. For $\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} = 0$, those metrics define a family of (2,2) Osserman manifolds. Moreover, they satisfy the duality principle for non-null vectors, but they don't satisfy the duality principle for null vectors. For example, the vector E_3 is null and its Jacobi operator $\mathcal{K}_{E_3} = 0$, but $\mathcal{K}_{E_1}(E_3) = -\frac{1}{2}\,\frac{\partial f_1}{\partial x_2}\,E_4$ (see, page 66 of [9]), it means that the duality principle for null vectors is not satisfied.

References

[1] V. Andrejić, Z. Rakić, On the duality principle in pseudo-Riemannian Osserman manifolds, *Jour. Geom. Phys.,* **57**, No. 10, (2007), 2158 – 2166.

[2] M. Baros and A. Romero, Indefinite Kähler manifolds, *Math. Ann.* **261**, (1982), 55 – 62.

[3] N. Blažić, N. Bokan, P. Gilkey, A note on Osserman Lorentzian manifolds, *Bull. London Math. Soc.* **29** (1997), 227 – 230.

[4] N. Blažić, N. Bokan, Z. Rakić, Osserman pseudo-Riemannian manifolds of signature $(2,2)$, *J. Aust. Math. Soc.* **71** (2001), 367 – 395.

[5] N. Blažić, N. Bokan, Z. Rakić, A note on Osserman conjecture and isotropic covariant derivative of curvature, *Proc. Amer. Math. Soc.* **128** (2000), 245 – 253.

[6] A. Bonome, R. Castro, E. Garcia-Rio, L. Hervella, R. Vazquez-Lorenzo, Nonsymmetric Osserman Indefinite Kahler Manifolds, *Proc. Amer. Math. Soc.* **126**, No. 9, (1998), 2763 – 2769.

[7] Q. S. Chi, A curvature characterization of certain locally rank-one symmetric spaces, *J. Diff. Geom.*, **28** (1988), 187 – 202.

[8] P. M. Gadea, J. M. Masqué, Classification of non-flat para-Kählerian space forms, *Houston J. Math.* **21**, No. 1, (1995), 89 – 94.

[9] E. Garcia-Rio, D. N. Kupeli, R. Vázquez-Lorenzo, *Osserman manifolds in semi-Riemannian geometry*, Lect. Notes Math. **1777**, Springer-Verlag, Berlin, Heidelberg, New York, 2002.

[10] E. Garcia-Rio, D. N. Kupeli, E. Vázquez-Abal, On a problem of Osserman in Lorentzian geometry, Differential Geom. Appl. **7**, (1997), 85100.

[11] E. Garcia-Rio, P. B. Gilkey, E. Vázquez-Abal, R. Vázquez-Lorenzo, Four-dimensional Osserman Metrics of Neutral Signature, 11 pages, `ArXiv: 0804.0436v1`.

[12] P. Gilkey, *Geometric Properties of Natural Operators Defined by the Riemannian Curvature Tensor*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

[13] Y. Nikolayevsky, Osserman manifolds of dimension 8, *Manuscripta Math.* **115** (2004), 31 – 53.

[14] Y. Nikolayevsky, Osserman conjecture in dimension $\neq 8, 16$, *Math. Ann.* **331** (2005), 505 – 522.

[15] Y. Nikolayevsky, On Osserman manifolds in dimension 16, *Proc. Conference Contemporary, Geometry and Related Topics, June 26 – July 2, 2005, Belgrade*, to appear (2006).

[16] A. I. Malcev, *Foundations of linear algebra*, Russian edituon, Nauka Moskva, 1975, Russian.

[17] R. Osserman, Curvature in the eighties, *Amer. Math. Monthly* **97** (1990), 731 – 756.

[18] R. Osserman, P. Sarnak, A new curvature ivariant and entropy of geodesic flows, *Invent. Math.* **77** (1984), 455 – 462.

[19] Z. Rakić, An example of rank two symmetric Osserman space, *Bull. Austral. Math. Soc.* **56** (1997), 517 – 521.

[20] Z. Rakić, On duality principle in Osserman manifolds, *Lin. Alg. Appl.* **296** (1999), 183 – 189.

[21] J. A. Wolf, *Spaces of constant curvature*, Univ. California, Berkeley, 1972.