Beyond Spacetime: On the Clifford Algebra Based Generalization of Relativity *

Matej Pavšič†
Jožef Stefan Institute, Jamova 39, Ljubljana, SLOVENIA

Abstract

Unification of fundamental interactions based on the theory of strings and other extended objects (branes), although very promising, has encountered certain technical and conceptual difficulties. Amongst other, it is not clear what is a deeper geometric principle beyond the string and brane theory. We point out that the physics of branes can be formulated in an infinite dimensional brane space whose points are kinematically possible branes. Although branes are systems with infinite number of degrees of freedom, it is possible to sample them with a finite number of degrees of freedom and formulate the theory in the so called Clifford space (C-space). This is a 16-dimensional space of oriented r-dimensional areas for $r = 0, 1, 2, 3, 4$. All dimensions of C-space have a direct physical meaning, which is a significant advantage in comparison to the usual string theories and Kaluza-Klein theories, where the extra dimensions have to be "compactified" in order to be unobservable. We assume that C-space has in general non vanishing curvature. The connection and vielbein of Clifford space are determined by the solutions to the generalized Einstein equations, and contain, not only the 4-dimensional gravitational field, but also other gauge fields. This enables a unification of interactions similar to that in Kaluza-Klein theories.

1. Introduction

The persisting problems such as quantum gravity and the unification of interactions suggest that our notion of spacetime should be revised. A promising new direction has been pioneered in refs [1]–[4]. The idea is that the arena for physics is a multidimensional configuration space associated with a physical system. In particular, we can consider a system of point particles. This has been investigated in detail within the context of (quantum) gravity by Barbour [1]. A somewhat different approach to the concept

* Work supported by the Slovenian Research Agency.
† e-mail address: matej.pavsic@ijs.si

of configuration space associated with a system of point particles has been considered in refs. [3, 4], where a generalization to extended objects, such as strings and branes has also been studied.

1.1. Configuration space for infinite dimensional objects – branes

Theories of strings and higher dimensional objects—branes—are very promising in explaining the origin and interrelationship of the fundamental interactions, including gravity [5, 6].

But there is a cloud. A question arises as to what is the geometric principle behind string and brane theories, and how to formulate them in a background independent way [7].

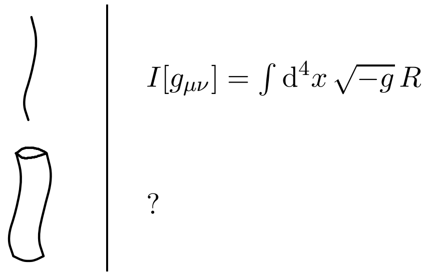

Figure 1: To a point particle there corresponds the Einstein-Hilbert action in spacetime. What is a corresponding space and action for a closed string?

Since such a fundamental issue has been left unsettled in the course of the development of string theory, it is not difficult to imagine that the latter theory is not yet finished. Recent serious criticism of string theory refers to an incomplete theory [8]. In the following we will consider the possibility that string/brane theories should take into account the concept of configuration space.

A brane can be considered as a point in an infinite dimensional space $\mathcal{M}$ with coordinates

$$X^\mu(\xi^a) \equiv X^{\mu(\xi)} \equiv X^M \tag{1}$$

where $X^\mu(\xi^a)$, $\mu = 0, 1, 2, , N-1$; $a = 0, 1, 3, , n-1$, $n < N$, are brane's embedding fuctions [9, 10]. This includes classes of tangentially deformed branes, which we can interpret as being physically different objects, not just as being related by reparametrizations of the brane's world manifold [9] (see Fig. 2).
All such objects are represented by different points of $\mathcal{M}$-space. The latter space is the configuration space associated with a brane. This is the space of all (infinitely many) possible configurations of a brane.

Instead of one brane we can take a 1-parameter family of branes $X^\mu(\tau, \xi^a)$ $\equiv X^{\mu(\xi)}(\tau) \equiv X^M(\tau)$, i.e., a curve (trajectory) in $\mathcal{M}$. In principle every

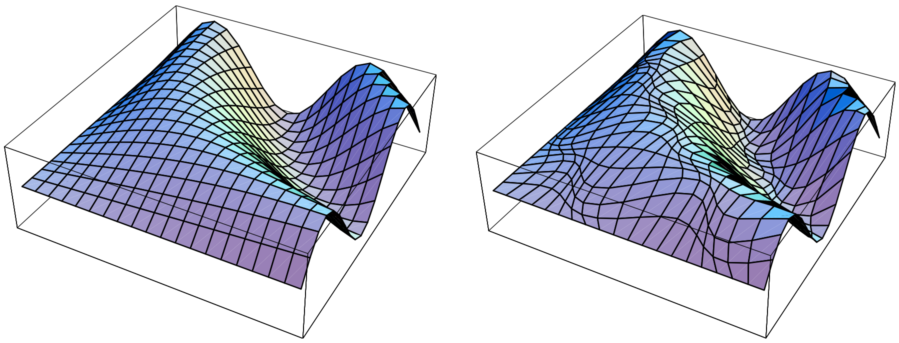

Figure 2: An example of a tangentially deformed membrane. Mathematically the surface on the left is the same as the surface on the right. Physically the two surfaces are different.

trajectory is kinematically possible. A particular dynamical theory then selects which amongst those kinematically possible branes and trajectories are dynamically possible. We assume that dynamically possible trajectories are *geodesics* in $\mathcal{M}$ determined by the minimal length action [9, 10]:

$$I[X^M] = \int \mathrm{d}\tau \; (\rho_{MN}\,\dot{X}^M \dot{X}^N)^{(1/2)} \tag{2}$$

Here ρ_{MN} is the metric of $\mathcal{M}$.

In particular, if metric is

$$\rho_{MN} \;\equiv\; \rho_{\mu(\xi')\nu(\xi'')} \;=\; \kappa\, \frac{\sqrt{|f(\xi')|}}{\sqrt{\dot{X}^2\,(\xi')}}\, \delta(\xi' - \xi'')\, \eta_{\mu\nu} \tag{3}$$

where $f_{ab} \equiv \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$ is the induced metric on the brane, $f \equiv \det\, f_{ab}$, $\dot{X}^2 \equiv \dot{X}^\mu \dot{X}^\nu g_{\mu\nu}$, ($\eta_{\mu\nu}$ being the Minkowski metric of the embedding spacetime), then the equations of motion derived from (2) are precisely those of a Dirac-Nambu-Goto brane [9, 10].

In this theory we assume that the metric (3) is just one particular choice amongst many other possible metrics of $\mathcal{M}$. But dynamically possible metrics are not arbitrary. We assume that they must be solutions of the Einstein equations in $\mathcal{M}$ [9, 10].

We take the brane space $\mathcal{M}$ as an arena for physics. The arena itself is a part of the dynamical system, it is not prescribed in advance. The theory is thus background independent. It is based on the geometric principle which has its roots in the brane space $\mathcal{M}$.

In summary, the infinite dimensional brane space $\mathcal{M}$ has in principle any metric that is a solution to the Einstein's equations in $\mathcal{M}$. For the particular diagonal metric (3) we obtain the ordinary branes, including strings. But it remains to be checked whether such particular metric is a solution of this

$$I[g_{\mu\nu}] = \int \mathrm{d}^4x \sqrt{-g}\, R$$

$$I[\rho_{\mu(\phi)\nu(\phi')}] = \int \mathcal{D}X \sqrt{|\rho|}\, \mathcal{R}$$

Figure 3: Brane theory is formulated in $\mathcal{M}$-space. The action is given in terms of the $\mathcal{M}$-space curvature scalar $\mathcal{R}$. We use abbreviation $\phi \equiv \phi^A = (\tau, \xi^a)$.

generalized dynamical system at all. If not, then this would mean that the ordinary string and brane theory is not exactly embedded into the theory based on a dynamical $\mathcal{M}$-space. The proposed theory goes beyond that of the usual strings and branes. It resolves the problem of background independence and the geometric principle behind the string theory (Fig. 3). Geometric principle behind the string theory is based on the concept of brane space $\mathcal{M}$, i.e., the configuration space for branes. Occurrence of gauge and gravitational fields in string theories is also elucidated. Such fields are due to string configurations. They occur in the expansion of a string state functional in terms of the Fock space basis. This can now be understood as well within the classical string theory based on the action (2) with $\mathcal{M}$-space metric ρ_{MN}, which is dynamical and satisfies the Einstein equations in $\mathcal{M}$. Multidimensionality of ρ_{MN} allows for extra gauge interactions, besides gravity. In the following we will point out how in the infinite dimensional space $\mathcal{M}$ one can factor out a finite dimensional subspace.

2. Finite dimensional description of extended objects

The Earth has a huge (practically infinite) number of degrees of freedom. And yet, when describing the motion of the Earth around the Sun, we neglect them all, except for the coordinates of *the centre of mass.*

Instead of infinitely many degrees of freedom associated with an extended object, we may consider *a finite number of degrees of freedom.*

2.1. The space of r-volumes

Strings and branes have infinitely many degrees of freedom. But at first approximation we can consider just *the center of mass* (Fig. 4a).

Next approximation is in considering the holographic coordinates $X^{\mu\nu}$ of the *oriented area* enclosed by the string (Fig. 4b).

We may go further and search for eventual thickness of the object. If the string has finite thickness, i.e., if actually it is not a string, but a 2-brane, then there exist the corresponding volume degrees of freedom $X^{\mu\nu\rho}$ (Fig. 5).

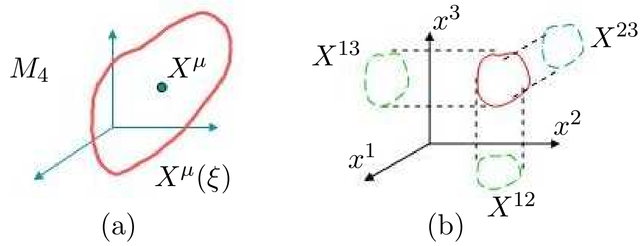

Figure 4: With a closed string one can associate the center of mass coordinates (a), and the area coordinates (b).

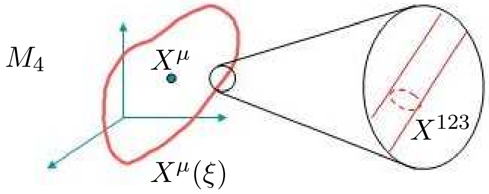

Figure 5: Looking with a sufficient resolution one can detect eventual presence of volume degrees of freedom.

In general, for an extended object in M_4, we have 16 coordinates

$$X^M \equiv X^{\mu_1 \ldots \mu_r}, \qquad r = 0, 1, 2, 3, 4 \tag{4}$$

They are projections of r-dimensional volumes (areas) onto the coordinate planes.

Oriented r-volumes can be elegantly described by Clifford algebra [11]. Let us illustrate this on the example of a 2-surface Σ bounded by a loop B. An infinitesimal surface element is given by the wedge product of two infinitesimal vectors $\mathrm{d}\xi_1$ and $\mathrm{d}\xi_2$, expanded in terms of the basis tangent vectors e_a, $a = 1, 2$:

$$\mathrm{d}\Sigma = \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2 = \mathrm{d}\xi_1^a \mathrm{d}\xi_2^b \, e_a \wedge e_b = \frac{1}{2} \, \mathrm{d}\xi^{ab} e_a \wedge e_b \tag{5}$$

where $\mathrm{d}\xi^{ab} = \mathrm{d}\xi_1^a \mathrm{d}\xi_2^b - \mathrm{d}\xi_2^a \mathrm{d}\xi_1^b$. Inserting the relation $e^a = \partial_a X^\mu \gamma_\mu$ between the basis vector e_a tangent to the surface Σ and the basis vectors γ_μ of the embedding space(time), and integrating over $\mathrm{d}\Sigma$, we have

$$\int_{\Sigma_B} \mathrm{d}\Sigma = \frac{1}{2} X^{\mu\nu} \gamma_\mu \wedge \gamma_\nu \tag{6}$$

where

$$X^{\mu\nu} = \frac{1}{2} \int_{\Sigma_B} \mathrm{d}\xi^{ab} (\partial_a X^\mu \partial_b X^\nu - \partial_a X^\nu \partial_b X^\mu) \tag{7}$$

which by the Stokes theorem gives

$$X^{\mu\nu} = \frac{1}{2} \oint_{\Sigma_B} \mathrm{d}s \left(X^\mu \frac{\partial X^\nu}{\partial s} - X^\nu \frac{\partial X^\mu}{\partial s} \right) \tag{8}$$

Here $X^\mu(s)$ are embedding functions of the boundary loop B, s being a parameter along the loop. Eq. (8) tells us that there is a mapping

$$X^\mu(s) \to X^{\mu\nu} \tag{9}$$

from infinite dimensional objects $X^\mu(s)$, describing loops, into the finite dimensional objects $X^{\mu\nu}$.

The above arrangement can describe two physically distinct situations:

(i) A loop B can be a closed string. Then $X^{\mu\nu}$ are bivector coordinates associated with the closed string.

(ii) A surface Σ can correspond to an open 2-brane whose boundary is B. Then $X^{\mu\nu}$ are bivector coordinates associated with the open 2-brane.

Analogous setup holds for objects of arbitrary dimensions:

$$X^{\mu_1\mu_2\ldots\mu_3} = \frac{1}{2} \int_{B_r} \mathrm{d}\xi^{a_1\ldots a_r}\, \partial_{[a_1} X^{\mu_1} \ldots \partial_{a_r]} X^{\mu_r} \tag{10}$$

2.2. Clifford space as a generalization of spacetime

the usual relativity, formulated in spacetime in which the interval is

$$\mathrm{d}s^2 \;=\; \eta_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu \tag{11}$$

one can consider the theory in which the interval is extended to the space of r-volumes, called pandimensional continuum [12] or Clifford space [13]–[17]:

$$\mathrm{d}S^2 \;=\; G_{MN}\, \mathrm{d}x^M \mathrm{d}x^N \tag{12}$$

Coordinates of Clifford space can be used to model extended objects [13, 15]. They are a generalization of the concept of center of mass. Instead of describing an extended object in "full detail", we can describe it in terms of the center of mass, area and volume coordinates. In particular, the extended object can be a fundamental string/brane.

Dynamics. Taking also a time like parameter τ, our object can be described by 16 functions $X^M(\tau)$. Let the action for an extended object described in terms of the coordinates of Clifford space be

$$I = \int \mathrm{d}\tau\, (G_{MN} \dot{X}^M \dot{X}^N)^{1/2} \tag{13}$$

If $G_{MN} = \eta_{MN}$ is Minkowski metric, then the equations of motion are

$$\ddot{X}^M \;\equiv\; \frac{\mathrm{d}^2 X^M}{\mathrm{d}\tau^2} \;=\; 0 \tag{14}$$

They hold for tensionless branes. For the branes with tension one has to replace η_{MN} with a generic metric G_{MN} with non vanishing curvature. Eq. (14) then generalizes to the corresponding geodesic equation

$$\frac{1}{\sqrt{\dot{X}^2}}\left(\frac{\dot{X}^M}{\sqrt{\dot{X}^2}}\right) + \Gamma^M_{JK}\frac{\dot{X}^J\dot{X}^K}{\dot{X}^2} = 0 \tag{15}$$

As an example let us consider the Dirac membrane, described by four embedding functions of three parameters $\xi^a = (\tau, \vartheta, \varphi)$:

$$X^\mu(\xi^a) \;=\; (X^0,\; r\,\sin\,\vartheta\,\cos\,\phi,\; r\,\sin\,\vartheta\,\sin\,\phi,\; r\,\cos\vartheta) \tag{16}$$

The metric on the worldsheet swept by the membrane is

$$\gamma_{ab} \;=\; \begin{pmatrix} \dot{X}_0^2 - \dot{r}^2 & 0 & 0 \\ 0 & -\,r^2 & 0 \\ 0 & 0 & -\,r^2\sin^2\,\vartheta \end{pmatrix} \tag{17}$$

The action is

$$I = \int \mathrm{d}\tau\,\mathrm{d}\vartheta\,\mathrm{d}\phi\,\sqrt{|\gamma|} = \int \mathrm{d}\tau\,4\pi r^2\sqrt{\dot{X}_0^2 - \dot{r}^2} \tag{18}$$

where $\sqrt{|\gamma|} \equiv \sqrt{|\det\gamma|} = \sqrt{\dot{X}_0^2 - \dot{r}^2}\,r^2\sin\vartheta$. Variation of the above action with respect to r and $X^0 = X_0$ gives the following equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{r}}{\sqrt{\dot{X}_0^2 - \dot{r}^2}}\right) + \frac{2\dot{X}_0^2}{r\sqrt{\dot{X}_0^2 - \dot{r}^2}} \;=\; 0 \tag{19}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{r^2\,\dot{X}_0}{\sqrt{\dot{X}_0^2 - \dot{r}^2}}\right) \;=\; 0 \tag{20}$$

If we now introduce the new variable according to eq. (10)

$$\begin{aligned} X^{123} &= \tfrac{1}{3!}\int \mathrm{d}r\,\mathrm{d}\vartheta\,\mathrm{d}\phi\;\partial_{[a}X^1\partial_b X^2\partial_{c]}X^3 = \tfrac{4\pi r^3}{3} \\ \dot{X}^{123} &= 4\pi r^2\dot{r} \\ \tfrac{dX^{123}}{dS} &= \tfrac{\dot{X}^{123}}{4\pi r^2\sqrt{\dot{X}_0^2-\dot{r}^2}} = \tfrac{\dot{r}}{\sqrt{\dot{X}_0^2-\dot{r}^2}} \end{aligned} \tag{21}$$

where $\mathrm{d}S = \;\mathrm{d}\tau\,4\pi r^2\sqrt{\dot{X}_0^2 - \dot{r}^2}$, then the equation of motion (19) becomes

$$\frac{\mathrm{d}^2X^{123}}{\mathrm{d}S^2} + \frac{2}{3X^{123}}\left(1 + \left(\frac{\mathrm{d}X^{123}}{\mathrm{d}S}\right)^2\right) = 0 \tag{22}$$

whereas eq. (20), due to the reparametrization invariance of our action (18), is redundant.

The equation of motion (22) can be considered as the geodesic equation (15) derived from the C-space action (13) for the case of a subspace described by two coordinates $X^M = (X^0, X^{123})$ with the metric

$$G_{MN} = \begin{pmatrix} C\tilde{X}^{4/3} & 0 \\ 0 & -1 \end{pmatrix} \tag{23}$$

where C is a constant, and where we have denoted $\tilde{X} \equiv X^{123}$. Namely, if we insert the particular metric (23) into the equation of geodesic (15), then we obtain eq. (22).

We can show that the above C-space description is equivalent to the Dirac membrane by directly comparing the actions. In the 2-dimensional subspace with coordinates $X^M = (X^0, X^{123})$, $X^{123} \equiv \tilde{X}$, and the metric (23) we have the following line element

$$\begin{aligned} \mathrm{d}S^2 &= G_{00}(\mathrm{d}X^0)^2 + G_{\tilde{X}\tilde{X}}\mathrm{d}\tilde{X}^2 \\ \mathrm{d}S^2 &= C\tilde{X}^{4/3}(\mathrm{d}X^0)^2 - \mathrm{d}\tilde{X}^2 \end{aligned} \tag{24}$$

Using

$$\begin{aligned} &\tilde{X} = \tfrac{4\pi r^3}{3}\,, \qquad \mathrm{d}\tilde{X} = 4\pi r^3 \mathrm{d}r \\ &\tilde{X}^{4/3} = \left(\tfrac{4\pi}{3}\right)^{4/3} r^4 \\ &C\left(\tfrac{4\pi}{3}\right)^{4/3} = (4\pi)^2 \end{aligned} \tag{25}$$

we have

$$\mathrm{d}S^2 = (4\pi r^2)^2\left(\mathrm{d}(X^0)^2 - \mathrm{d}r^2\right) \tag{26}$$

Inserting the latter line element into the action

$$I[X^M] = \int \mathrm{d}S = \int d\tau\,(G_{MN}\dot{X}^M\dot{X}^N)^{1/2} \tag{27}$$

we obtain

$$I = \int \mathrm{d}\tau\,(4\pi r^2)^2\sqrt{(\dot{X}^0)^2 - \dot{r}^2} \tag{28}$$

which is the action for the Dirac membrane.

The above example explicitly shows why description of branes with non vanishing tension requires non trivial metric of the brane configuration space.

Such higher dimensional configuration space, associated with branes, enables unification of fundamental interactions à la Kaluza-Klein [18]. For alternative, although related approaches see [19].

3. Curved Clifford space

3.1. On the general relativity in C-space

Clifford space is a straightforward generalization of spacetime manifold M. In the spacetime manifold $M_{1,3}$, in which we choose a point $\mathcal{P}_0$ as the origin, the *vectors* $x = x^\mu \gamma_\mu|_{\mathcal{P}_0} \in V_{1,3}(\mathcal{P}_0) = T_{\mathcal{P}_0}(M_4)$ are in one-to-one correspondence with other points $\mathcal{P} \in M_4$, at least within a region $B \subset M_4$ around $\mathcal{P}_0$.

Similarly, the *polyvectors* $X = X^M \gamma_M|_{\mathcal{E}_0} \in \mathcal{C}\ell(1,3)$ are in one-to-one correspondence with the points $\mathcal{E}$ of a 16-dimensional manifold C, called a *Clifford space*, in which we choose a point $\mathcal{E}_0$ as the as the origin, such that $\gamma_M|_{\mathcal{E}_0} \in T_{\mathcal{E}_0}(C)$ are tangent polyvectors at $\mathcal{E}_0$. In other words, vectors of the tangent space at $\mathcal{E}_0 \in C$ can be mapped into the points $\mathcal{E} \in C$ of a region around $\mathcal{E}_0$. A tangent space to a point $\mathcal{E} \in C$ is the vector space defined by the Clifford algebra $\mathcal{C}\ell(1,3)$.

Coordinate basis elements

$$\gamma_M \equiv \gamma_{\mu_1 \ldots \mu_r} \ , \quad r = 0,1,2,3,4 \tag{29}$$

depend on position $\mathcal{E} \in C$, and are not defined as a wedge product.

Orthonormal basis elements

$$\gamma_A = \gamma_{a_1 a_2 \ldots a_r} = \gamma_{a_1} \wedge \gamma_{a_2} \wedge \ldots \wedge \gamma_{a_r} \tag{30}$$

are defined as the wedge product of the orthonormal basis vectors $\gamma_a \in V_{1,3}$.

The two sets of basis elements are related in terms of the C-space vielbein:

$$\gamma_M = e_M{}^A \gamma_A. \tag{31}$$

From the last relation it is obvious that γ_M, in general, has indefinite grade. It may happen that at a point $\mathcal{E}_1$ we have $\gamma_M = \gamma_{\mu_1} \wedge \ldots \wedge \gamma_{\mu_r}$, but because of eq. (31) this, in general, does no longer hold at some other point $\mathcal{E}_2$.

Whereas the metric of C-space is

$$\gamma_M^\ddagger * \gamma_N = g_{MN} \tag{32}$$

the metric of the tangent space spanned by γ_A is the generalized Minkowski metric:

$$\gamma_A^\ddagger * \gamma_B = \eta_{AB} \tag{33}$$

The operation $\ddagger$ reverses the order of vectors in a Clifford product:

$$(\gamma_{a_1} \gamma_{a2} \ldots \gamma_{a_r})^\ddagger = \gamma_{a_r} \ldots \gamma_{a_2} \gamma_{a1}.$$

Here $A * B \equiv \langle AB \rangle_0$ is the scalar product between polyvectors A and B, where the subscript 0 denotes the zero grade part, i.e., the scalar part.

Besides the basis elements γ_M and γ_A, we can define the reciprocal elements $\gamma^M,\ \gamma^A$ by the relations

$$(\gamma^M)^\ddagger * \gamma_N = \delta^M{}_N \ , \quad (\gamma^A)^\ddagger * \gamma_B = \delta^A{}_B \tag{34}$$

Let us introduce the differential operator $\partial_M \equiv \partial_{\gamma_M}$, called *derivative*[1], acting on polyvector valued fields.

If it acts on a *scalar* field ϕ , then it behaves just as a partial derivative:

$$\partial_M \phi = \frac{\partial \phi}{\partial x^M} \tag{35}$$

If it acts on a coordinate frame field or on an orthonormal field, then it defines the corresponding connections:

$$\partial_M \gamma_N = \Gamma^J_{MN} \gamma_J \tag{36}$$
$$\partial_M \gamma_A = -\Omega_A{}^B{}_M \gamma_B \tag{37}$$
$$\tag{38}$$

Derivative of a generic polyvector field $A = A^N \gamma_N$ is

$$\partial_M (A^N \gamma_N) = \partial_M A^N \gamma_N + A^N \partial_M \gamma_N = (\partial_M A^N + \Gamma^N_{MK} A^K)\gamma_N \equiv \mathrm{D}_M A^N \, \gamma_N \tag{39}$$

where $\mathrm{D}_M A^N \equiv \partial_M A^N + \Gamma^N_{MK} A^K$ are components of the covariant derivative in the coordinate basis, i.e., the 'covariant derivative' of the tensor analysis. Here A^N are *scalar* components of A, and $\partial_M A^N$ is just the partial derivative of a scalar field with respect to X^M:

$$\partial_M \equiv \left(\frac{\partial}{\partial s}, \ \frac{\partial}{\partial x^{\mu_1}}, \ \frac{\partial}{\partial x^{\mu_1 \mu_2}}, ..., \ \frac{\partial}{\partial x^{\mu_1 \dots \mu_n}} \right) \tag{40}$$

Other symbols for the derivative used in the literature are $\Box_M$, ∇_M, D_{γ_M}, ∇_{γ_M}.

We define the curvature of C-space in the analogous way as in the ordinary spacetime, namely by employing the commutator of the derivatives [11, 9, 16]. Using eq. (36) we have

$$[\partial_M, \partial_N]\gamma_J = R_{MNJ}{}^K \gamma_K \tag{41}$$

where

$$R_{MNJ}{}^K = \partial_M \Gamma^K_{NJ} - \partial_N \Gamma^K_{MJ} + \Gamma^R_{NJ}\Gamma^K_{MR} - \Gamma^R_{MJ}\Gamma^K_{NR} \tag{42}$$

[1] This operator is a generalization to curved C-space of the derivative ∂_μ which acts in an n-dimensional space V_n, and was defined by Hestenes [11] (who used a different symbol, namely $\Box_\mu$).

is the curvature of C-space. Using (41) and (34) we can express the curvature according to

$$(\gamma^K)^\ddagger * ([\partial_M, \partial_N]\gamma_J) = R_{MNJ}{}^K \tag{43}$$

An analogous relation we have if the commutator of the derivatives operates on orthonormal basis elements and use eq. (37):

$$[\partial_M, \partial_N]\gamma_A = R_{MNA}{}^B \gamma_B \tag{44}$$

where

$$R_{MNA}{}^B = -(\partial_M \Omega_A{}^B{}_N - \partial_N \Omega_A{}^B{}_M + \Omega_A{}^C{}_N \Omega_C{}^B{}_M - \Omega_A{}^C{}_M \Omega_C{}^B{}_N) \tag{45}$$

In the theory considered here, the concept of spacetime should be replaced by that of C-space. Spacetime is just a start. From its basis we can build a larger space —Clifford space—which also is physical. It has 16 dimensions, and therefore can serve as a realization of Kaluza-Klein theory, without the need to introduce extra dimensions of the underlying spacetime M_4.

Let us assume that the classical action contains a term which describes the "point particle" in C-space and a kinetic term which describes the dynamics of the C-space itself:

$$I[X^M, G_{MN}] = \int \mathrm{d}\tau \, (\dot{X}^M \dot{X}^N G_{MN})^{1/2} + \frac{\kappa}{16\pi} \int [\mathrm{d}x] R \tag{46}$$

Here $\dot{X}^M \equiv \mathrm{d}X^M/\mathrm{d}\tau$, κ a constant, $[\mathrm{d}x] \equiv \mathrm{d}^{16}x$ the measure on C-space, and $R = R_{MNJ}{}^N$ the curvature scalar (see eqs. (41)–(45)) of C-space, analogous to the curvature scalar of the ordinary general relativity. The action is invariant under local (pseudo) rotations in tangent space $T_X(C)$, and under general coordinate transformations in C-space.

Variation of the action (46) with respect to X^M gives the geodetic equation (15) in C-space, whereas variation with respect to G_{MN} gives the C-space Einstein's equations:

$$R^{MN} - \frac{1}{2} G^{MN} R = 8\pi\kappa \int \mathrm{d}\tau \, \delta^{(C)}(x - X(\tau)) \dot{X}^M \dot{X}^N \tag{47}$$

where $\delta^{(C)}$ is the δ-function in C-space.

When looking from the 4-dimensional spacetime, the equation of geodesic (15) contains besides the usual gravitation also other interactions. They are encoded in the metric components G_{MN} of C-space. Gravity is related to the components $G_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$, while the gauge fields due to other interactions are related to the components $G_{\mu\bar{M}}$, where the index $\bar{M} \neq \nu$ assumes 12 possible values, excluding the four values $\nu = 0, 1, 2, 3$. In addition, there are also interactions due to the components $G_{\bar{M}\bar{N}}$, but they have not the property of the ordinary Yang-Mills fields.

The occurrence of gauge interaction within the framework of a higher dimensional space is a well known feature of Kaluza-Klein theories. But in our approach the "extra dimensions" of the 16-dimensional Clifford space need not be compactified, because they are associated with observable properties of physical objects, namely their effective r-volumes, as explained in sec. 2 and refs. [15].

In quantizing such classical theory, based on the action (13) which describes 'point particle' in C, one expects to be able to proceed as usual, and arrive at the quantum field theory in C. So there would be no necessity to increase the dimensionality of (4-dimensional) spacetime, from which we start in building the 16-dimensional Clifford space C.

Consider now the possibility of choosing to describe the object of Figs. 4,5, not with a set of coordinates $X^M = \{X, X^{\mu_1}, X^{\mu_1\mu_2}, X^{\mu_1\mu_2,\mu_3}, X^{\mu_1\mu_2,\mu_3,\mu_4}\}$, but rather employ a more detailed description. Let ξ be a parameter along the "centroid" loop, i.e., a closed string of Fig. 4a, and let $X^\mu(\xi)$ be its embedding functions in M_4. But if we look more closely (Fig. 5), we find that at every value of parameter ξ the string has a thickness: At every value of ξ, instead of a point there may be a loop, described by $X^{\mu_1\mu_2}$. If we look even closer, we may find even more structure, encoded in coordinates $X^{\mu_1\mu_2,\mu_3}$ and $X^{\mu_1\mu_2,\mu_3,\mu_4}$. Altogether, taking into account as well a time like parameter τ, our object is described by 16 functions $X^M(\tau,\xi)$. These are functions describing the embedding of a string's worldsheet into a 16-dimensional Clifford space C. Instead of the usual string worldsheet $X^\mu(\tau,\xi)$, $\mu = 0,1,2,3$, we have a worldsheet in a higher dimensional space which is not spacetime, but Clifford space (i.e., a configuration space) C. It was shown that such string living in C—which happens to have signature $(8,8)$—can be be consistently quantized [20] by employing the Jackiw definition of vacuum state [21, 22].

Hence, there is a possibility of having a consistent string theory without employing extra dimensions of spacetime, provided that one does not consider infinitely thin strings, but allows for their thickness, encoded in the coordinates of Clifford space.

4. Summary

We have considered a theory in which spacetime is replaced by a larger space, namely the configuration space associated with a system under consideration. For strings/branes the configuration space is infinite dimensional, but it can be reduced to a corresponding finite dimensional space, the so called Clifford space C. Since configuration space has extra dimensions, its metric provides description of additional interactions, beside the 4-dimensional gravity, just as in Kaluza-Klein theories. In this theory there is no need for extra dimensions of *spacetime*. The latter space is a subspace of the *configuration space* C, and all dimensions of C are physical. Therefore, there is no need for a compactification of the extra dimensions of C.

We have presented some theoretical justifications for the insight that the basic space of physics could be associated with configurations of physical systems. Search for possible realistic solutions to the theory and their comparison with observations is a project worth of being pursued in the future.

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