Singularities of Smooth Maps and General Notion of Curvature*

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Abstract

We introduce a new notion of a curvature of a superconnection, different from the one defined by a purely algebraic analogy with the curvature of a linear connection. This new notion comes from the study of singularities of smooth sections of vector bundles. We demonstrate that the classical examples of obstructions to a local equivalence of sections: exterior differential, Riemannian tensor, Weil tensor, curvature of a linear connection and Nijenhuis tensor can be treated in terms of one general approach. This approach, applied to the superconnection leads to a simpler and more natural notion of a curvature of a superconnection.

1. A Brief Review of the Notion of a Superconnection

In this section we give a brief review of the notions of a superconnection and of the corresponding supercurvature introduced by Quillen [8].

By $\xi = (E, p, M)$ we denote a vector bundle over the manifold M ($\dim M = m$, by ξ^* - the dual bundle and by $C^\infty(\xi)$ - the space of the smooth sections

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of ξ. $\Omega^k(M) = C^\infty(\Lambda^k T^*(M))$ is the space of the differential k-forms on the manifold M and

$$\Omega(M) = \overset{m}{\underset{k=0}{\oplus}} \Omega^k(M) = C^\infty(\overset{m}{\underset{k=0}{\oplus}} \Lambda^k T^*(M))$$

is the space of the nonhomogeneous forms on M. $\Omega^k(\xi) = C^\infty(\Lambda^k T^*(M) \otimes \xi)$ is the space of the differential k-forms with values in ξ and $\Omega(\xi) = \oplus_{k=0}^m \Omega^k(\xi)$ - the space of the nonhomogeneous forms with values in ξ. A linear connection ∇ on ξ is a linear differential operator $\nabla : \Omega^0(\xi) \longrightarrow \Omega^1(\xi)$ with the property $\nabla(f.\psi) = \mathrm{d}f \otimes \psi + f\nabla(\psi)$, $\psi \in \Omega^0(\xi)$, $f \in \Omega^0(M)$.

The covariant differential $\mathrm{d}^\nabla : \Omega^k(\xi) \longrightarrow \Omega^{k+1}(\xi)$ is defined by the property $\mathrm{d}^\nabla(\alpha \otimes \psi) = \mathrm{d}\alpha \otimes \psi + (-1)^k \mathrm{d} \wedge \nabla(\psi)$, $\alpha \in \Omega^k(M)$, $k = 1, 2, ..., m$ and $\psi \in \Omega^0(\xi)$. Its square $\mathrm{d}^\nabla \circ \mathrm{d}^\nabla : \Omega^k(\xi) \longrightarrow \Omega^{k+2}(\xi)$ is an $\Omega^0(M)$-linear operator and $F^\nabla = \mathrm{d}^\nabla \circ \mathrm{d}^\nabla : \Omega^0(\xi) \longrightarrow \Omega^2(\xi)$ is the curvature of ∇. In local coordinates$(\nabla(\psi))^a_\mu = \partial_\mu \psi^a + A^a_{\mu b}\psi^b$ and $F^{\nabla}{}_{\mu\nu}{}^a{}_b = (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])^a_b$.

The upper notions have an algebraic analogy in the case of a $\mathbb{Z}_2$ graded super vector bundle [8]. Let $\xi = \xi_+ \oplus \xi_-$ be a $\mathbb{Z}_2$ graded vector bundle over the manifold M. Then the dual bundle $\xi^* = \xi^*_+ \oplus \xi^*_-$ is also $\mathbb{Z}_2$ graded in a natural way. The induced $\mathbb{Z}_2$ grading in $\Omega(\xi)$ is given by

$$\Omega(\xi) = \Omega(\xi)_+ \oplus \Omega(\xi)_- \,,$$

where $\Omega(\xi)_+ = \oplus_k(\Omega^{2k}(\xi_+) \oplus \Omega^{2k+1}(\xi_-))$ and $\Omega(\xi)_- = \oplus_k(\Omega^{2k+1}(\xi_+) \oplus \Omega^{2k}(\xi_-))$. Let ∇ be a linear connection on the vector bundle ξ compatible with the $\mathbb{Z}_2$ grading of ξ, $\nabla : \Omega^0(\xi_\pm) \longrightarrow \Omega^1(\xi_\pm)$.

Let $\chi \in \Omega^0(\xi^* \otimes \xi)$ be an odd tensor field, i.e. $\chi(x) : \xi_{x\pm} \longrightarrow \xi_{x\mp}$ (or $\chi(x) \in (\xi^* \otimes \xi)_{x-}$). By definition (proposed by Quillen) a superconnection ∇_s is the odd linear operator $\nabla_s = (\nabla + \chi) : \Omega^0(\xi)_\pm \longrightarrow \Omega^{0,1}(\xi)_\mp$, where $\Omega^{0,1}(\xi) = \Omega^0(\xi) \oplus \Omega^1(\xi)$. It is easy to see that the superconnection ∇_s has the property $\nabla_s(f.\psi) = df \otimes \psi + f\nabla_s(\psi)$, $f \in \Omega^0(M)$, $\psi \in \Omega^0(\xi)$. In more details, if we write $\psi = \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}$, where ψ_+ and ψ_- are the even and the odd parts of the super vector field ψ respectively, then

$$\nabla_s \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = \begin{bmatrix} \nabla & \chi_{+-} \\ \chi_{-+} & \nabla \end{bmatrix} \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = \begin{bmatrix} \nabla\psi_+ + \chi_{+-}\psi_- \\ \nabla\psi_- + \chi_{-+}\psi_+ \end{bmatrix}.$$

The covariant differential $\mathrm{d}^{\nabla_s} : \Omega^k(\xi) \longrightarrow \Omega^{k,k+1}(\xi)$ is defined by a purely algebraic analogy with the "classical" covariant differential d^∇.

$$\mathrm{d}^{\nabla_s}(\alpha \otimes \psi) = \mathrm{d}\alpha \otimes \psi + (-1)^k \alpha \wedge \nabla_s(\psi)\,;\ \alpha \in \Omega^k(M)\,,\ \psi \in \Omega^0(\xi)\,.$$

$F^{\nabla_s} = \mathrm{d}^{\nabla_s} \circ \mathrm{d}^{\nabla_s} : \Omega^0(\xi) \longrightarrow \Omega^{0,1,2}(\xi)$ is an even differential operator of order zero, i.e. F^{∇_s} is a tensor, $F^{\nabla_s} \in \Omega^{0,1,2}(\xi^* \otimes \xi)$,

$$F^{\nabla_s} = \chi^2 + \mathrm{d}^{\nabla}(\chi) + F^{\nabla} . \tag{1}$$

The nonhomogeneous tensor F^{∇_s} defined by an algebraic analogy with the "classical" case is called "supercurvature" of the superconnection ∇_s.

In the classical case the connection ∇ is connected with the parallel transport and the curvature is an obstruction to the flatness of the parallel transport. In other words the curvature of a connection is an obstruction to its local equivalence with the flat connection. In the "super" case there are no natural notions of a parallel transport and of a flat superconnection. But we can look for an obstruction to the local equivalence of two superconnections. To motivate the idea we consider in the next section some classical examples of obstructions to a local equivalence of sections of some bundles as particular examples of a general scheme.

2. A List of Obstructions to a Local Equivalence

2.1. An Obstruction for an Arbitrary Nondegenerate Differential 2-form to be Diffeomorphic to the Canonical Symplectic Form

Let M be a smooth even-dimensional manifold, $\dim M = m = 2l$ and let $\omega_0 \in \Omega^2(M)$. In local coordinates $\omega_0 = \sum_{\mu=1}^{l} \mathrm{d}x^\mu \wedge \mathrm{d}x^{l+\mu}$. Every diffeomorphism $\varphi \in Diff(M)$ has a natural action on $\Omega^2(M)$. For every $\omega \in \Omega^2(M)$ we have $\varphi^*(\omega)_{\mu\nu}(x) = \frac{\partial\varphi^\alpha}{\partial x^\mu}(x)\frac{\partial\varphi^\beta}{\partial x^\nu}(x)\omega_{\alpha\beta}(\varphi(x))$. Does there exist (at least locally) $\varphi \in Diff(M)$ such that $\varphi^*(\omega) = \omega_0$ in the case $\det\{\omega_{\mu\nu}(x)\} \neq 0$? An obstruction to this is the condition $\mathrm{d}\omega \neq 0$. If $\mathrm{d}\omega = 0$ then there exists (at least locally) $\varphi \in Diff(M)$ such that $\varphi^*(\omega) = \omega_0$. The 3-form $\mathrm{d}\omega$ is in some sense the curvature of the 2-form ω.

2.2. An Obstruction for an Euclidean Metric to be in a Canonical Form

Let M be a smooth manifold, $g_0 = \sum_{\mu=1}^{m} \mathrm{d}x^\mu \otimes \mathrm{d}x^\mu$ - an Euclidean metric on M in a canonical form and g - an arbitrary Euclidean metric on M. Does there exist (at least locally) $\varphi \in Diff(M)$ such that

$$\varphi^*(g)_{\mu\nu}(x) = \frac{\partial\varphi^\alpha}{\partial x^\mu}(x)\frac{\partial\varphi^\beta}{\partial x^\nu}(x)g_{\alpha\beta}(\varphi(x)) = g_{0\mu\nu}(x) = \delta_{\mu\nu} .$$

The Riemannian tensor $R(g) \neq 0$ is an obstruction to this. If $R(g) = 0$ then there exists (at least locally) $\varphi \in Diff(M)$ such that $\varphi^*(g) = g_0$. The Riemannian tensor $R(g)$ is the curvature of the metric g.

2.3. An Obstruction for an Euclidean Metric to be Conformally Equivalent to the Canonical Metric

Let g_0 and g be the same as in Subsection 2.2. Does there exist (at least locally) $\varphi \in Diff(M)$ such that $\varphi^*(g)(x) = f(x)g_0(x)$, $f(x) \neq 0$. The Weil tensor

$$W(g) = R(g) - \frac{1}{m-2}(Ric(g) - \frac{1}{2(m-2)}r(g).g)\underline{\triangle} g \neq 0\,,$$

where $Ric(g)$ is the Ricci tensor and $r(g)$ - the scalar curvature, is an obstruction to this [2]. If $W(g) = 0$ then there exists (at least locally) $\varphi \in Diff(M)$ such that $\varphi^*(g)(x) = f(x)g_0(x)$. The Weil tensor $W(g)$ is the conformal curvature of the metric g.

2.4. An Obstruction for a Given Connection to be Gauge Equivalent to the Flat One

Let $\xi = (E, p, M)$ be a vector bundle over the manifold M and ∇ - a linear connection on ξ . Let $\varphi \in Aut^V(\xi)$ be a vertical automorphism of ξ. The automorphism φ has a natural action on the space of the linear connections. In local coordinates

$$\varphi^*(\nabla)_\mu = \partial_\mu + \varphi^*(A)_\mu\,,\ \varphi^*(A)_\mu(x) = \varphi^{-1}(x).A_\mu(x).\varphi(x) + \varphi^{-1}(x).\partial_\mu\varphi(x)\,.$$

Can we find $\varphi \in Aut^V(\xi)$ satisfying the condition $\varphi^*(A)_\mu = 0$? The curvature tensor $\{F_{\mu\nu}\} \neq 0$ is an obstruction to this. If $F_{\mu\nu} = 0$ we can find (at least locally) an automorphism $\varphi \in Aut^V(\xi)$ such that

$$\varphi^*(A)_\mu(x) = \varphi^{-1}(x).A_\mu(x).\varphi(x) + \varphi^{-1}(x).\partial_\mu\varphi(x) = 0\,.$$

2.5. An Obstruction for a Given Almost Complex Structure to be Diffeomorphic to the Canonical One

Let M be an even dimensional smooth manifold, $\dim M = m = 2l$ and let

$$J_0 = \sum_{i=1}^{l}\left(\mathrm{d}x^i \otimes \frac{\partial}{\partial x^{2+i}} - \mathrm{d}x^{l+i} \otimes \frac{\partial}{\partial x^i}\right)$$

be an almost complex structure on M in a canonical form. Let J be an arbitrary almost complex structure on M [4], $J \in C^\infty(T^*(M) \otimes T(M))$ and $J^2(x) = -\mathbf{1}$. In the same coordinates $J(x) = J^\mu_\nu(x)\mathrm{d}x^\nu \otimes \frac{\partial}{\partial x^\mu}$. Every diffeomorphism $\varphi \in Diff(M)$ has a natural action on $C^\infty(T^*(M) \otimes T(M))$. For every $J \in C^\infty(T^*(M) \otimes T(M))$ we have

$$\varphi^*(J)^\mu_\nu(x) = \frac{\partial\varphi^{-1\mu}}{\partial x^\alpha}(\varphi(x))\frac{\partial\varphi^\beta}{\partial x^\nu}(x)J^\alpha_\beta(\varphi(x))\,.$$

Does there exist $\varphi \in Diff(M)$ such that $\varphi^*(J) = J_0$? The non-vanishing Nijenhuis tensor,

$$N(J)^{\rho}_{\mu\nu} = J^{\alpha}_{\mu}\partial_{\alpha}J^{\rho}_{\nu} - J^{\alpha}_{\nu}\partial_{\alpha}J^{\rho}_{\mu} - J^{\rho}_{\alpha}\partial_{\mu}J^{\alpha}_{\nu} + J^{\rho}_{\alpha}\partial_{\nu}J^{\alpha}_{\mu}$$

is an obstruction to this. If $N(J) = 0$ then there exists (at least locally) $\varphi \in Diff(M)$ such that $\varphi^*(J) = J_0$ (see Newlander - Nirenberg theorem [5]). The Nijenhuis tensor can be written as a vector-valued 2-form

$$N(J)(X,Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]\,.$$

The tensor $N(J)$ is in some sense the curvature of the almost complex structure J.

All the obstructions to the local equivalence of sections of the corresponding fibre bundles we mentioned can be viewed as "curvatures" in the broad sense of the word. We can consider them as particular examples of a general notion of a curvature considered as an universal obstruction to the local equivalence in terms of jet bundles and algebraization of the differential operators. This scheme applied to the case of a superconnection leads to a notion of a supercurvature different from the one, obtained by a purely algebraic analogy. We consider the definition given in Section 3 to be the adequate one.

3. General Notion of Curvature

Let $\xi = (E, p, M)$ be a vector bundle. The adopted coordinates (x^{μ}, u^a) on ξ induce the corresponding coordinates $(x^{\mu}, u^a, u^a_{\nu}, u_{\nu_1\nu_2}, ..., u^a_{\nu_1,...,\nu_k})$, $1 \le \nu_1 \le \nu_2 \le ... \le \nu_j \le m$, $j = 2, 3, ..., k$ on the k-th jet bundle $j^k(\xi)$ of ξ, $j^k(\xi) = (E^k, p^k, M)$ for every $k = 1, 2, 3, ...$ ([7], [9]). Every element of the fibre $j^k(\xi)_x$ of $j^k(\xi)$ equals to $j^k(\psi(x))$ - the k-jet of some $\psi \in C^{\infty}(\xi)$ at the point x. In coordinates $u^a(j^k(\psi(x)) = \psi^a(x)$, $u^a_{\nu_1...\nu_j}(j^k(\psi(x))) = \partial_{\nu_1}...\partial_{\nu_j}\psi^a(x)$, where ψ^a are the components of the field ψ. The mapping $C^{\infty}(\xi) \longrightarrow C^{\infty}(j^k(\xi))$, given by

$$\psi(x) \longmapsto (\psi^a(x), \partial_{\mu}\psi^a(x), ..., \partial_{\mu_1}...\partial_{\mu_k}\psi^a(x))$$

is the so called jet lifting of the section ψ and plays the role of an universal differential operator of order k ([7], [9]).

Let H be a "functional" group and $F_h : E \longrightarrow E$, $h \in H$ be a fibre preserving action of the group H on the bundle $\xi = (E, p, M)$. For sake of simplicity we will have in mind only the group $Aut^V(\xi)$ of vertical automorphisms of ξ, or $Diff(M)$ - the group of diffeomorphisms of M. In the coordinates (x^{μ}, u^a) the action F of the group H reads $F_h(x^{\mu}, u^a) = (\varphi^{\mu}_h(x), F^a_h(x, u))$, where $h \in H$, $\varphi_h : M \longrightarrow M$ is an action of the group H on the base M of the bundle.

The group H has a natural action $\tilde{F}$ on $C^\infty(\xi)$ given by

$$\tilde{F}_h(\psi)(x) = F_h(\psi(\varphi_h^{-1}(x)))\,, \tag{2}$$

where $\psi \in C^\infty(\xi)$, $h \in H$, $x \in M$. The action (2) of the group H on $C^\infty(\xi)$ induces an action of H on the jet-bundle $j^k(\xi)$ for every $k = 1, 2, 3, ...$([1], [3]). Let ψ_1 and ψ_2 be two sections of the bundle ξ. The problem we deal with is the (local) equivalence of ψ_1 and ψ_2 with respect to the group H, i.e. the existence of an element $h \in H$ such that

$$\tilde{F}_h(\psi_2) = \psi_1\,. \tag{3}$$

In other words for given ψ_1 and ψ_2, the expression (3) is an equation for $h \in H$. If the equation (3) is satisfied then its k-jet lifting

$$\tilde{F}_h(j^k(\psi_2)) = j^k(\psi_1) \tag{4}$$

is also satisfied for every k. Inversely, if we can proof that for some k the equation (4) has no solution for $h \in H$ then obviously this is an obstruction to the solvability of (3), i.e. to the local equivalence of ψ_1 and ψ_2. The equation (4) involves the derivatives of ψ_1 and ψ_2 up to the order k. Therefore it is easier to deal with (4) then with (3).

We begin with the study of the condition imposed by the equation (4) on the k-jets of ψ_1 and ψ_2 at a point $x_0 \in M$.

Let $G_{x_0} := \{h \in H \mid \varphi_h(x_0) = x_0\}$ be the stationary group of the point x_0. The group G_{x_0} has a natural action on $j^k(\xi)_{x_0}$ for every k. Let us first consider the case $k = 0$. The space $j^0(\xi)_{x_0}$ is simply the fibre ξ_{x_0}. For $h \in G_{x_0}$ the equation (4) leads to

$$\tilde{F}_h(\psi_2(x_0)) = \psi_1(x_0)\,. \tag{5}$$

Insolvability of (5) with respect to h means that $\psi_1(x_0)$ and $\psi_2(x_0)$ belong to different orbits of G_{x_0} in ξ_{x_0}. If the fibre ξ_{x_0} is a homogeneous space of the group G_{x_0} there is no obstruction to the solvability of (4) arising from the equation (5). Then looking for an obstruction we go to $k = 1$:

$$\tilde{F}_h(j^1(\psi_2)_{x_0}) = j^1(\psi_1)_{x_0}\,. \tag{6}$$

If the fibre $j^1(\xi)_{x_0}$ is again a homogeneous space of the group G_{x_0} there is no an obstruction to the solvability of equation (4) arising from the equation (6). We proceed to the first k for which the fibre $j^k(\xi)_{x_0}$ is not a homogeneous space of the group G_{x_0}. Let $\tilde{\pi}_{x_0}$ be the canonical projection to the factor space $j^k(\xi)_{x_0}/G_{x_0}$

$$\tilde{\pi}_{x_0} : j^k(\xi)_{x_0} \longrightarrow j^k(\xi)_{x_0}/_{G_{x_0}}\,. \tag{7}$$

If

$$\tilde{\pi}_{x_0}(j^k(\psi_2)_{x_0}) \neq \tilde{\pi}_{x_0} j^k(\psi_1)_{x_0}) \tag{8}$$

then the k-jets $j^k(\psi_2)_{x_0}$ and $j^k(\psi_1)_{x_0}$ belong to different orbits of the group G_{x_0} and the equation

$$\tilde{F}_h(j^k(\psi_2))_{x_0} = j^k(\psi_1)_{x_0} \tag{9}$$

has no solution with respect to $h \in G_{x_0}$, which is an obstruction to the local equivalence of the sections ψ_2 and ψ_1 in a neighborhood of the point x_0. This consideration is valid for every point $x \in M$ and we obtain a fibre preserving map (over the identity)

$$\tilde{\pi} : j^k(\xi) \longrightarrow j^k(\xi)/H\,,$$

where the fibre of $j^k(\xi)/H$ at a point $x \in M$ is the factor space $j^k(\xi)_x/G_x$. The fibre preserving map $\tilde{\pi}$ is the symbol of the differential operator

$$\pi : C^\infty(\xi) \longrightarrow C^\infty(j^k(\xi)/H)\,,$$

i.e. $\pi(\psi) = \tilde{\pi}(j^k(\psi))$. The condition $\pi(\psi_2) \neq \pi(\psi_1)$ is an obstruction to the (local) equivalence of ψ_1 and ψ_2. Therefore we consider the differential operator π as a curvature - the general notion of curvature that we propose in this paper ($\pi(\psi)$ is the curvature of the section ψ).

4. Explicit expression for the projection π and the new notion of curvature of superconnection

One can show that the classical results listed in Section 2 are explicit examples of the differential operator π. Namely:

2.1. The sections we consider are the two-forms ω on M, the functional group is the group $Diff(M)$, $k = 1$ and $\pi(\omega) = \mathrm{d}\omega$.

2.2. The sections we consider are the metrics g on M, the functional group is the group $Diff(M)$, $k = 2$ and $\pi(g) = R(g)$ - the Riemannian curvature tensor.

2.3. The sections we consider are the conformally equivalent classes $[g]$ of metrics on M, the functional group is $Diff(M)$, $k = 2$ and $\pi([g]) = W(g)$ - the Weil conformal tensor.

2.4. The sections we consider are the linear connections ∇ on a vector bundle ξ, the functional group is the group $Aut^V(\xi)$, $k = 1$ and $\pi(\nabla) = F^\nabla$ - the Yang-Mills curvature tensor.

2.5. The sections we consider are the almost complex structures J on M, the functional group is $Diff(M)$, $k = 1$ and $\pi(J) \sim N(J)$ - the Nijenhuis tensor.

We will briefly describe the obtaining of the projection π for the examples 2.3 - 2.5 and we will apply the general notion of a curvature to the case of a superconnection.

Finding the curvature operator (the projection π) comes to the following problem of linear algebra: We have vector space K (coming from the jet

$j_0^k(\xi)$), a vector space K_0 (coming from parameters of the stationary group), and a linear map $i : K_0 \longrightarrow K$. The action of the stationary group on K has the form $u \longrightarrow u + i(v)$, $\forall u \in K$, $\forall v \in K_0$. The projection π is the canonical projection $K \longrightarrow K/_{i(K_0)}$ on the factor space and we have the following exact sequence

$$0 \longrightarrow K_0 \overset{\overset{c}{\longleftarrow}}{\underset{}{\xrightarrow{i}}} K \xrightarrow{\pi} K/_{i(K_0)} \longrightarrow 0\,. \tag{10}$$

If $c : K \longrightarrow K_0$ is a linear map such that the operator $A = c \circ i$ is invertible then $K = i(K_0) \oplus \ker(c)$, $i \circ A^{-1} \circ c : K \longrightarrow i(K_0)$ is the projection on the image $i(K_0)$ and $\ker(c)$ is isomorphic to the factor space $K/_{i(K_0)}$. The operator

$$\mathbf{1} - i \circ A^{-1} \circ c : K \longrightarrow \ker(c) \cong K/_{i(K_0)} \tag{11}$$

gives the projection π we are looking for. For more details see [6].

In the case of a linear connection on a vector bundle (Example 2.4) the exact sequence (10) corresponds to

$$0 \rightarrow S^2(L^*) \otimes End(\mathbb{R}^n) \overset{\overset{c}{\longleftarrow}}{\xrightarrow{i}} L^* \otimes L^* \otimes End(\mathbb{R}^n) \xrightarrow{\pi} \Lambda^2(L^*) \otimes End(\mathbb{R}^n) \rightarrow 0.$$

Here K is the space of all bilinear matrix-valued two-forms, K_0 - the space of all symmetric bilinear matrix-valued two-forms and i is the natural embedding. (L is the standard fibre of the tangent bundle $T(M)$ of the base M.) The factor space is the space of the antisymmetric bilinear forms. The projection π is obviously the antisymmetrization operator and it is exactly the Yang-Mills curvature.

In the case of Weil tensor (Example 2.3) the exact sequence (10) corresponds to

$$0 \longrightarrow S^2(L^*) \overset{\overset{c}{\longleftarrow}}{\xrightarrow{i}} \widetilde{S^2(\Lambda^2 L^*)} \xrightarrow{\pi} \ker(c) \longrightarrow 0.$$

Here K is the space of all algebraic curvature tensors - all bilinear symmetric two-forms defined on $\Lambda^2 L$ that considered as four-forms on L are antisymmetric (satisfying the Bianchi identity)[2]. K_0 is the space of all symmetric two-forms on L. The map i is defined by $i(r) = r \underline{\triangle} g$, where $\underline{\triangle}$ is the Kulkarni - Nomizu product [2], $r \in S^2(L^*)$, $g \in S^2(L^*)$ is fixed - it is the value of the metric at the point x_0. In this example the map c is the Ricci map, $c(R) = Ric(R)$ ($Ric(R)_{\mu\nu} = R_{\mu\alpha\nu\beta} g^{\alpha\beta}$). Then $A(r) = c \circ i(r) = Ric(r \underline{\triangle} g) = (m-2)r + tr(r)g$. It is easy to check that

$$A^{-1}(r) = \frac{1}{m-2} r - \frac{1}{2(m-2)(m-1)} tr(r) g\,,$$

$$i \circ A^{-1} \circ c(R) = (\frac{1}{m-2} Ric(R) - \frac{1}{2(m-2)(m-1)} tr(Ric(R)).g) \underline{\triangle} g$$

is the Schouten tensor. For the projection π we obtain

$$\begin{aligned}\pi(R) &= (\mathbf{1} - i \circ A^{-1} \circ c)(R) \\ &= R - (\tfrac{1}{m-2} Ric(R) - \tfrac{1}{2(m-2)(m-1)} tr(Ric(R)).g) \triangle g\,.\end{aligned}$$

As a differential operator on Euclidean metrics on M this is exactly the Weil tensor $W(g) = R(g) - \frac{1}{m-2}(Ric(g) - \frac{1}{2(m-1)}\mathbf{r}(g))\triangle g$, where $\mathbf{r}(g)$ is the scalar curvature of the metric g.

In the case of Nijenhuis tensor (Example 2.5) the sequence (10) corresponds to

$$S^2(L^*) \otimes L \underset{i}{\overset{\xleftarrow{c}}{\longrightarrow}} W \xrightarrow{\pi} \ker(c) \longrightarrow 0.$$

The vector space K is the subspace $W = \{B \in L^* \otimes L^* \otimes L \mid JB(u,v) + B(J(u),v) = 0, u,v \in L\} \subset L^* \otimes L^* \otimes L$, J is the value of the almost complex structure at the point x_0, K_0 is the vector space of all symmetric bilinear two forms with values in L and the map i is defined by

$$i(B)(u,v) = JB(u,v) - B(Ju,v)\,.$$

For the map c we take the symmetrization

$$c(B)(u,v) = \frac{1}{2}(B(v,u) + B(u,v))$$

and we obtain

$$\begin{aligned}(i \circ A^{-1} \circ c)(B)(u,v) &= \frac{1}{2}(B(u,v) + B(v,u) + B(Ju,Jv) - B(Jv,Ju))\,, \\ \pi(B)(u,v) &= (\mathbf{1} - i \circ A^{-1} \circ c)B(u,v) \\ &= \frac{1}{2}(B(u,v) - B(v,u) - JB(u,Jv) + JB(v,Ju))\,.\end{aligned}$$

As a differential operator on J

$$\pi(J) = -\frac{1}{2}J.N(J)\,,$$

where $N(J)$ is the Nijenhuis tensor. The almost complex structure J is non-degenerate ($J^2 = -1$) so $\pi(J)$ is equivalent to $N(J)$. (Formally $N(J)$ is not a projection operator.)

The consideration of the case of superconnection is very similar to the consideration of the case of linear connection but due to the $\mathbb{Z}_2$ grading there are much more indices. In this case the projection π leads to the tensor [6]

$$\tilde{F}^{\nabla_s} = (\chi, \nabla(\chi), F^{\nabla})\,,$$

which is a simpler expression then $(\chi^2, \nabla(\chi), F^{\nabla})$ (see(1)) obtained by purely algebraic analogy with the curvature of a linear connection. This leads to more acceptable expression for the Lagrangians of the models of interacting particles in the supersymmetrical field theories which contain the square of the supercurvature.

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