Cohomological analysis of renormalization on configuration spaces*

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Abstract

A cohomological analysis of the renormalization freedom is performed in the Epstein-Glaser scheme on a flat Euclidean space. We study the deviation from commutativity between the renormalization and the action of all linear partial differential operators. It defines a Hochschild 1–cocycle and the renormalization ambiguity corresponds to a *nonlinear* subset in the cohomology class of this renormalization cocycle. We have shown that the related cohomology spaces can be reduced to de Rham cohomologies of the so called "(ordered) configuration spaces". We have also found cohomological differential equations that exactly determine the renormalization cocycles up to the renormalization freedom. This analysis is a first step towards a new approach for computing renormalization group actions. It can be also naturally extended to manifolds as well as to the case of causal perturbation theory.

1 Introduction

Perturbation theory in Quantum Field Theory (QFT) is one of the technically most difficult subjects in the contemporary theoretical physics. This is, first and foremost, due to the appearance of complicated integrals in higher orders, as well as, to the complexity of the accompanying renormalization. For the realistic QFT models there are practically no numerical results for arbitrary orders in perturbation theory. There are also a few methods that allow to perform calculations to all orders. Without pretending to give justice to various approaches to the subject we just point out the general analysis of perturbative renormalization in recent work of Connes-Kreimer (see e.g., [9, 3, 4]).

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The present work is a first step to a new approach for determining the action of the renormalization group in perturbative QFT (i.e., for calculating beta functions). It offers in addition a geometric insight to the problem. The general idea of the method is to perform a cohomological analysis of the renormalization ambiguity and to use it to determine the renormalization group action. Furthermore, we have separated the problem from the particular models of perturbative QFT, i.e. we consider all possible theories and even more general situations. It is this generality that makes the geometric interpretation possible. It is also important that we do not confine our treatment to the one parameter action of the renormalization group but consider all linear partial differential operators. This is done in order to restrict as much as possible the related cohomology owing to the general properties of the algebra of all differential operators. Our geometric view favors the study of renormalization in "coordinate space". This approach has been originally developed by Bogolubov, and Epstein and Glaser [5] on Minkowski space and recently applied to more general pseudo-Riemann manifolds (see e.g., [2, 8]). It is also called causal perturbation theory. This approach has a simpler counterpart in Euclidean QFT [7]. We choose to work here within this Euclidean framework, and even on $\mathbb{R}^D$, but our analysis can be extended to manifolds, as well as to the case of the causal perturbation theory on pseudo-Riemann manifolds. Our choice was motivated by the fact that the geometric structures appearing in the analysis are much more transparent in the Euclidean approach.

2 Motivation from perturbative QFT

In perturbative *euclidean* Quantum Field Theory (QFT) one computes the Green functions as formal power series of the type:

$$\begin{aligned} & G_N(\mathrm{z}_1,\dots,\mathrm{z}_N;\underline{g};\{R_n\}) \\ & = \sum_{\underline{r}\geqslant 0} \int\limits_{\mathbb{R}^{D|\underline{r}|}} d^D\mathrm{x}_1\cdots d^D\mathrm{x}_{|\underline{r}|}\, \frac{\underline{g}^{\underline{r}}}{\underline{r}!}\, R_{|\underline{r}|}\, \mathcal{A}_{\underline{r}}(\mathrm{x}_1,\dots,\mathrm{x}_{|\underline{r}|};\mathrm{z}_1,\dots,\mathrm{z}_N)\,. \end{aligned} \tag{2.1}$$

Here: $\mathrm{x}_k = (x_k^1,\dots,x_k^D)$ and $\mathrm{z}_k \in \mathbb{R}^D$; $\underline{r} = (r_1,\dots,r_s)$, $|\underline{r}| = \sum_j r_j$ and $\underline{r}!$ $= \prod_j r_j!$ are multiindex notations; $\underline{g} = (g_1,\dots,g_s)$ is a system of coupling constants; $\mathcal{A}_{\underline{r}}(\mathrm{x}_1,\dots,\mathrm{x}_{|\underline{r}|};\mathrm{z}_1,\dots,\mathrm{z}_N)$ are Feynman amplitudes (sums over Feynman graphs) with $|\underline{r}|$ internal and N external vertices; $\{R_n\}$ is a system of renormalization maps to be defined later. The important point for us in Eq. (2.1) is that after smearing the external points z_k of the Feynman amplitudes $\mathcal{A}_{\underline{r}}(\mathrm{x}_1,\dots,\mathrm{x}_{|\underline{r}|};\mathrm{z}_1,\dots,\mathrm{z}_N)$ by test functions they become finite sums of products of type:

$$\Bigg(\prod_{1\leqslant j<k\leqslant n} G_{jk}(\mathrm{x}_j-\mathrm{x}_k)\Bigg)\Bigg(\prod_{m=1}^{n} F_m(\mathrm{x}_m)\Bigg)\,, \tag{2.2}$$

where $G_{jk}(\mathrm{x}_j - \mathrm{x}_k)$ are "propagators" and $F_m(\mathrm{x}_m)$ are smooth functions on $\mathbb{R}^D$, which arise from the smearing of the external propagators ($F(\mathrm{x}) = \int G(\mathrm{x}-\mathrm{y})\ f(\mathrm{y})\ d^D\mathrm{y}$). Since the propagators $G_{jk}(\mathrm{x}_j - \mathrm{x}_k)$ are regular functions for $\mathrm{x}_j \neq \mathrm{x}_k$, the integrands of type (2.2) are well defined, regular functions on the subspace of all pairwise distinct arguments $(\mathrm{x}_1,\dots,\mathrm{x}_n) \in (\mathbb{R}^D)^{\times n} (\cong \mathbb{R}^{Dn})$. The latter subspace of $\mathbb{R}^{Dn}$ is also called an *ordered configuration space* over $\mathbb{R}^D$ and is denoted by $F_n(\mathbb{R}^D)$. The configuration spaces are generally introduced for arbitrary manifold (or set) X by:

$$F_n(X) \,=\, \Big\{ (\mathrm{x}_1,\dots,\mathrm{x}_n) \in X^{\times n} \,:\, \mathrm{x}_j \,\neq\, \mathrm{x}_k \ \text{ if } \ j \,\neq\, k \Big\}$$

and they are well studied (see e.g., [6]). Hence, the renormalization maps R_n in (2.1) are introduced in order to make the integrals well defined. In particular, R_n should extend smooth functions on configuration spaces to distributions over the whole space.[1] The system $\{R_n\}$ have to satisfy certain natural properties that we will consider in Sect. 3. Here we will generally explain the aim of our cohomological analysis.

A perturbative QFT is said to be renormalizable if after changing the system of renormalization maps, $\{R_n\} \to \{R'_n\}$, there exits a unique formal power series

$$\underline{\alpha}(\underline{g}) \,=\, \sum_{\underline{r}\,\geqslant 0} \underline{a}_{\underline{r}}\, \frac{\underline{g}^{\underline{r}}}{\underline{r}!}$$

such that

$$G_N(\mathrm{z}_1,\dots,\mathrm{z}_N;\underline{g};\{R'_n\}) \,=\, G_N(\mathrm{z}_1,\dots,\mathrm{z}_N;\underline{\alpha}(\underline{g});\{R_n\})$$

(in the sense of formal power series). In particular, one can change the renormalization maps by a dilation (i.e., changing the "scale"):

$$R_n^\lambda \,:=\, d_\lambda \circ R_n \circ d_\lambda^{-1}$$

(where $(d_\lambda f)(\mathrm{x}_1,\mathrm{x}_2,\dots) := f(\lambda^{-1}\mathrm{x}_1,\lambda^{-1}\mathrm{x}_2,\dots)$ for $\lambda \in \mathbb{R}^+$). As a result, this generates the action of renormalization group ($R^+ \ni \lambda$) on the coupling constants

$$\{R_n^\lambda\} \quad \to \quad \underline{\alpha}_\lambda(\underline{g})\,,$$

whose generator is the so called *beta function*

$$\underline{\beta}(\underline{g}) \,=\, \lambda\, \frac{d}{d\lambda}\, \underline{\alpha}_\lambda(\underline{g}) \Big|_{\lambda\,=\,1}\,.$$

[1] Let us point that we do not consider here the infrared problem, i.e., an additional extension related to the integration over an infinite volume (but it can be treated by the same method). This is because we shall be mainly interested in the renormalization group action (see below), which in QFT can be extracted only by its ultraviolet renormalization (or, its short distance properties).

Clearly, to compute the formal power series of $\underline{\beta}(\underline{g})$ one has to know the "commutators"

$$(\mathbf{x} \cdot \partial_{\mathbf{x}}) \circ R_n - R_n \circ (\mathbf{x} \cdot \partial_{\mathbf{x}}) \,,$$

where $\mathbf{x} \cdot \partial_{\mathbf{x}} := \sum\limits_{k,\mu} x_k^{\mu} \frac{\partial}{\partial x_k^{\mu}}$. We shall generalize this task and will look for the commutators

$$c_n[A] \,=\, A \circ R_n - R_n \circ A$$

for all linear partial differential operators A. It turns out that $c_n[A]$ is a certain Hochschild cocycle and changing the renormalization corresponds to adding a coboundary (see Sect. 4). So, we would like to find some cohomological equations that would determine $c_n[A]$.

From our analysis (see Sect. 3), it follows that the renormalization ambiguity allows us to achieve $c_n[f] = 0$ for smooth functions f (i.e., differential operators of zeroth order). This allows us to extend our methods also for manifolds without even any metric structure on them. Moreover, the remaining nontrivial part of the cocycle $c_n[\partial_{x_k^{\mu}}]$ ($\partial_{x_k^{\mu}} := \frac{\partial}{\partial x_k^{\mu}}$) can be characterized by certain cohomological equations (Eqs. (4.6) and (6.4)). We have proven in Theorem 4.1 that the ambiguity in the solutions of these equations exactly corresponds to the renormalization freedom. This is done by using de Rham cohomologies of configuration spaces. So, in the subsequent section we shall introduce the precise notion of renormalization maps. Then, we analyze the remaining nontrivial cohomological properties of the renormalization maps and their reduction to de Rham cohomologies.

3 Renormalization maps

We consider the problem of extending distributions from $F_n(\mathbb{R}^D)$ to the whole space $\mathbb{R}^{Dn}$ following the analog of the Epstein-Glaser scheme on Euclidean spaces. In this approach an important role plays the notion of scaling degree. Here we shall define it as follows. Let $u(\mathbf{x},\mathbf{y})$ be a distribution over $V \times U$ ($\ni (\mathbf{x},\mathbf{y})$), where V is an open cone in R^N and U is an open set in $\mathbb{R}^M$. The scaling degree of $u(\mathbf{x},\mathbf{y})$ with respect to the first vector argument is defined as:

$$\mathrm{sc.d.}_{(1)}\, u \,:=\, \inf \{\lambda \in \mathbb{R} \,:\, \underset{\varepsilon \downarrow 0}{\text{w-lim}}\; \varepsilon^{\lambda}\, u(\lambda \mathbf{x},\mathbf{y}) \,=\, 0\} \,.$$

Then we set

$$\begin{aligned} \mathscr{D}'_t(V \times U) \,:=& \{u(\mathbf{x}) \in \mathscr{D}'(V \times U) \,:\, \mathrm{sc.d.}_{(1)}\, u \,<\, \infty, \\ & \quad u \text{ has a finite order}\} \,, \\ \mathcal{C}^{\infty}_t(V \times U) \,:=& \,\mathcal{C}^{\infty}(V \times U) \cap \mathscr{D}'_t(V \times U) \end{aligned}$$

(we define $\mathcal{C}^{\infty}(X)$ as the complex vector space of all smooth functions $X \to \mathbb{C}$ over a real manifold X, which take *complex* values).

Let us also set for short

$$F_n := F_n(\mathbb{R}^D)$$

(an open cone in $(\mathbb{R}^D)^{\times n} \equiv \mathbb{R}^{Dn}$) and let

$$\widehat{\Delta}_n := \Big\{(\mathrm{x}_1,\dots,\mathrm{x}_n) \in \mathbb{R}^{Dn} \, : \, \mathrm{x}_j = \mathrm{x}_k \ \text{ for some } \ 1 \leqslant j < k \leqslant n\Big\}\,,$$
$$\Delta_n := \Big\{(\mathrm{x}_1,\dots,\mathrm{x}_n) \in \mathbb{R}^{Dn} \, : \, \mathrm{x}_1 = \dots = \mathrm{x}_n\Big\}\,,$$

be the rough and the total diagonals, respectively (and so we have the identity:

$$\mathbb{R}^{Dn} \setminus \widehat{\Delta}_n \equiv F_n\,).$$

The **renormalization maps** are defined as a system of linear maps

$$\mathcal{R}_{n,U} : \mathscr{D}'_t(F_n \times U) \to \mathscr{D}'(\mathbb{R}^{Dn} \times U)\,,$$

labelled by an integer $n \geqslant 2$ and an open set $U \subseteq \mathbb{R}^M$ for any $M \geqslant 0$ such that the following conditions are satisfied for every $u \in \mathscr{D}'_t(F_n \times U)$:

- $(r1)$ $\mathcal{R}_{n,U}(u)\big|_{F_n \times U} = u$ (i.e., $\mathcal{R}_{n,U}$ makes an extension of distributions).
- $(r2)$ $\mathrm{sc.d.}_{(1)}\, \mathcal{R}_{n,U}(u) \leqslant \mathrm{sc.d.}_{(1)}\, u$.
- $(r3)$ For every $h \in \mathcal{C}^\infty(\mathbb{R}^{Dn} \times U)$ we have $\mathcal{R}_{n,U}(h\,u) = h\,\mathcal{R}_{n,U}(u)$.
- $(r4)$ Let $S \subsetneqq \{1,\dots,n\}$, $m := |S| \geqslant 2$, $V_S := \{(\mathrm{x}_1,\,\dots,\,\mathrm{x}_n) \in \mathbb{R}^{Dn} : \mathrm{x}_j \neq \mathrm{x}_k$ if $j \in S$ and $k \notin S\}$ and finally, let $\chi \in \mathcal{C}^\infty(\mathbb{R}^{Dn})$ be such that $\operatorname{supp}\chi \subset V_S$; then we impose that the following equality holds:

$$\mathcal{R}_{n,U}(\chi\,u) \;=\; \mathcal{R}_{m,\,\mathbb{R}^{D(n-m)}\times U} \circ \mathcal{R}_{n-m,\,F_m\times U}(\chi\,u)\,. \tag{3.1}$$

  Condition (3.1) can be restated as follows. If we write the arguments of χu as $\mathbf{x}_S = (\mathrm{x}_s)_{s\,\in\,S}$, $\mathbf{x}_{S^c} = (\mathrm{x}_s)_{s\,\notin\,S}$, and $\mathbf{y} \in U$ then $\mathcal{R}_{n,U}(\chi\,u)$ gives the same result as if we first apply $\mathcal{R}_{n-m}$ with respect to $\mathbf{x}_{S^c}$ and then apply $\mathcal{R}_m$ with respect to $\mathbf{x}_S$.

Note that conditions $(r1)$–$(r3)$ are linear (inhomogeneous), while $(r4)$ is a nonlinear with respect to the whole system $\{\mathcal{R}_{n,U}\}$. We shall consider a system of renormalization maps $\{\mathcal{R}_{n,U}\}$ as defined by more simple objects, which we call **primary renormalization maps**. The latter are defined as linear maps

$$\mathcal{P}_{N,U} : \mathscr{D}'_t\Big((\mathbb{R}^N \backslash \{0\}) \times U\Big) \to \mathscr{D}'(\mathbb{R}^N \times U)\,,$$

labelled by an integer $N \geqslant 2$ and an open set $U \subseteq \mathbb{R}^M$ for any $M \geqslant 0$ such that for every $u \in \mathscr{D}'_t\Big((\mathbb{R}^N \backslash \{0\}) \times U\Big)$ the following conditions are satisfied:

($p1$) $\mathcal{P}_{N,U}(u)\big|_{(\mathbb{R}^N\backslash\{0\})\times U} = u$.

($p2$) $\mathrm{sc.d.}_{(1)}\,\mathcal{P}_{N,U}(u) \leqslant \mathrm{sc.d.}_{(1)}\,u$.

($p3$) For every $h \in \mathcal{C}^\infty(\mathbb{R}^N \times U)$ we have $\mathcal{P}_{N,U}(h\,u) = h\,\mathcal{P}_{N,U}(u)$.

For the sake of simplicity we shall often omit the second subscript U in the notations $\mathcal{R}_{n,U}$ and $\mathcal{P}_{N,U}$. (More complete construction of the systems of renormalization maps $\{\mathcal{R}_n\}$ and $\{\mathcal{P}_N\}$ can be found in [10].) We note that conditions ($r1$), ($r2$) and ($r4$), as well as ($p1$) and ($p2$) are a slight generalization of the usual construction due to Epstein and Glaser (originally, in the causal case). The generalization is at the point that we consider the renormalization maps as acting linearly on the whole space of distributions with finite scaling degree and not only to those distributions that have bounded degree, not exceeding some given number. More significant generalization of the usual Epstein–Glaser approach are conditions ($r3$) and ($p3$). They will play a very important role in our analysis in the subsequent sections.

For the sake of completeness we briefly sketch the construction of $\{\mathcal{R}_{n,U}\}$ by $\{\mathcal{P}_{n,U}\}$. For $n = 2$ noting that

$$F_2 \;\cong\; (\mathbb{R}^D\backslash\{0\}) \times \mathbb{R}^D$$

(for instance, under the map $(\mathrm{x}_1,\mathrm{x}_2) \mapsto (\mathrm{x}_1 - \mathrm{x}_2, \mathrm{x}_2)$) we can set

$$\mathcal{R}_{2,U} \;:=\; \mathcal{P}_{D,\,\mathbb{R}^D\times U}\,.$$

For $n > 2$ the sets V_S introduced in condition ($r4$) form an open covering of $\mathbb{R}^{Dn}\backslash\Delta_n$ and hence we have a uniquely determined map

$$\widehat{\mathcal{R}}_{n,U} : \mathscr{D}'_t(F_n \times U) \to \mathscr{D}'_t\Big((\mathbb{R}^{Dn}\backslash\Delta_n) \times U\Big)$$

such that $\widehat{\mathcal{R}}_{n,U}(u)\big|_{V_S\times U}$ is fixed by the right hand side of (3.1). Again we have:

$$\mathbb{R}^{Dn}\backslash\Delta_n \;\cong\; (\mathbb{R}^{D(n-1)}\backslash\{0\}) \times \mathbb{R}^D$$

(by $(\mathrm{x}_1,\dots,\mathrm{x}_n) \mapsto (\mathrm{x}_1 - \mathrm{x}_n,\dots,\mathrm{x}_{n-1} - \mathrm{x}_n)$) and hence, we can set

$$\mathcal{R}_{n,U} \;:=\; \mathcal{P}_{D(n-1),\,\mathbb{R}^D\times U} \circ \widehat{\mathcal{R}}_{n,U}\,. \tag{3.2}$$

As the singularities that we consider in the renormalization are only with respect to the differences $\mathrm{x}_j - \mathrm{x}_k$ we shall restrict further our analysis to functions that depend only on them. Let us introduce the reduced configuration space

$$\widetilde{F}_n \;:=\; \{(\mathrm{x}_1,\dots,\mathrm{x}_n) \in F_n : \mathrm{x}_k \neq 0\ (\forall k)\} \;\equiv\; F_n(\mathbb{R}^D\backslash\{0\})$$

and its complement in $\mathbb{R}^{Dn}$, the reduced rough diagonal,

$$\widetilde{\Delta}_n := \mathbb{R}^{Dn} \backslash \widetilde{F}_n .$$

We have projection

$$F_n \to \widetilde{F}_{n-1} : (\mathrm{x}_1, \dots, \mathrm{x}_n) \mapsto (\mathrm{x}_1 - \mathrm{x}_n, \dots, \mathrm{x}_{n-1} - \mathrm{x}_n) ,$$

which induces an embedding

$$\mathcal{C}_t^\infty(\widetilde{F}_{n-1}) \hookrightarrow \mathscr{D}_t'(F_n)$$

(the translation invariant functions) and we shall further restrict the renormalization maps to $\mathcal{C}_t^\infty(\widetilde{F}_{n-1})$ under this embedding:

$$\begin{aligned} R_n &:= \mathcal{R}_n \Big|_{\mathcal{C}_t^\infty(\widetilde{F}_{n-1})} : \mathcal{C}_t^\infty(\widetilde{F}_{n-1}) \to \mathscr{D}'(\mathbb{R}^{D(n-1)}) , \\ \widehat{R}_n &:= \widehat{\mathcal{R}}_n \Big|_{\mathcal{C}_t^\infty(\widetilde{F}_{n-1})} : \mathcal{C}_t^\infty(\widetilde{F}_{n-1}) \to \mathscr{D}'(\mathbb{R}^{D(n-1)} \backslash \{0\}) . \end{aligned} \tag{3.3}$$

Thus, Eq. (3.2) reads

$$R_n = \mathcal{P}_{D(n-1)} \circ \widehat{R}_n . \tag{3.4}$$

4 Cohomological analysis of renormalization maps

Let us introduce again the "commutators"

$$c_n[A] = A \circ R_n - R_n \circ A ,$$

for $A \in \mathfrak{D}_{D(n-1)}$, where $\mathfrak{D}_N$ is the associative algebra of all linear partial differential operators over $\mathbb{R}^N$. By the extension property $(r1)$, the image of each $c_n[A]$ lies in the vector space of all distributions supported at the (reduced) rough diagonal:

$$c_n[A] : \mathcal{C}_t^\infty(\widetilde{F}_{n-1}) \to \mathscr{D}'[\widetilde{\Delta}_{n-1}] .$$

Furthermore, $c_n[A]$ is a Hochschild 1–cocycle for the algebra $\mathfrak{D}_{D(n-1)}$ with coefficients in the bimodule of all linear maps $\mathcal{C}_t^\infty(\widetilde{F}_{n-1}) \to \mathscr{D}'[\widetilde{\Delta}_{n-1}]$: we have

$$A_1 \circ c_n[A_2] - c_n[A_1 \circ A_2] + c_n[A_1] \circ A_2 = 0 . \tag{4.1}$$

Clearly, the change the renormalization map $R_n \to R_n'$ will change the cocycle c_n by a coboundary:

$$c_n[A] - c_n'[A] = A \circ b - b \circ A ,$$

where $b = R_n - R'_n$. Nevertheless, it is not so simple to take the full renormalization ambiguity into account: as we mentioned in the previous section, condition $(r4)$ is nonlinear and hence, the class

$$\{\{c_n\} : \{c_n\} \text{ is generated by } \{R_n\}\}$$

will form a nonlinear subset in the direct sum of the cohomology classes of all c_n.

By the property $(r3)$ we have $c_n[x_k^\mu] = 0$ and thus, what remains to be determined (due to Eq. (4.1)) is

$$\omega_{n;\,k,\,\mu} := c_n[\partial_{x_k^\mu}] = [\partial_{x_k^\mu}, R_n] \tag{4.2}$$

$(\partial_{x_k^\mu} := \dfrac{\partial}{\partial x_k^\mu}, k = 1, \dots, n-1, \mu = 1, \dots, D)$. For short, we shall denote the pair of indices (k, μ) in Eq. (4.2) by a single index ξ (or η, ...) running from 1 to $D(n-1)$; then the corresponding components x_k^μ shall be denoted by x^ξ. Applying Eq. (3.4) to the definition of $\omega_{n;\xi}$ we obtain a decomposition

$$\begin{aligned} \omega_{n;\,\xi} &= \gamma_{n;\,\xi} + \widehat{\omega}_{n;\,\xi} \qquad (n > 2), \qquad \omega_{2;\,\xi} \equiv \gamma_{2;\,\xi}\,, && (4.3)\\ \gamma_{n;\,\xi} &:= [\partial_{x^\xi}, \mathcal{P}_{D(n-1)}] \circ \widehat{R}_n \qquad (n > 2)\,, && (4.4)\\ \widehat{\omega}_{n;\,\xi} &:= \mathcal{P}_{D(n-1)} \circ [\partial_{x^\xi}, \widehat{R}_n] \qquad (n > 2)\,. \end{aligned}$$

The maps $\gamma_{n;\,\xi}$ are simpler than $\omega_{n;\,\xi}$ since they take values that are distributions supported at the *reduced total diagonal*, i.e., at the origin $\mathbf{0} \in \mathbb{R}^{D(n-1)}$:

$$\gamma_{n;\,\xi} : \mathcal{C}_t^\infty(\widetilde{F}_{n-1}) \to \mathscr{D}'[\mathbf{0}] \tag{4.5}$$

(this is due to condition $(p1)$). On the other hand, the remaining part $\widehat{\omega}_{n;\,\xi}$ of $\omega_{n;\,\xi}$ in Eq. (4.3) is determined by the renormalization induction. So, the only new information is contained in $\gamma_{n;\,\xi}$.

The maps $\gamma_{n;\,\xi}$ satisfy the "differential equations":

$$\begin{aligned} &[\partial_{x^\xi}, \gamma_{2;\,\eta}] - [\partial_{x^\eta}, \gamma_{2;\,\xi}] = 0\,,\\ &[\partial_{x^\xi}, \gamma_{n;\,\eta}] - [\partial_{x^\eta}, \gamma_{n;\,\xi}] && (4.6)\\ &\quad = -[\partial_{x^\xi}, \mathcal{P}_{D(n-1)}] \circ [\partial_{x^\eta}, \widehat{R}_n] + [\partial_{x^\eta}, \mathcal{P}_{D(n-1)}] \circ [\partial_{x^\xi}, \widehat{R}_n] \quad (n > 2) \end{aligned}$$

for all $\xi, \eta = 1, \dots, D(n-1)$, which are derived by a straightforward computation. We shall characterize $\gamma_{n;\,\xi}$ by these equations. Before that, let us point out that *the right hand side of (4.6) is determined by the renormalization induction*. The reason is that the values of $[\partial_{x^\xi}, \widehat{R}_n]$ are distributions supported at the rough diagonal $\widetilde{\Delta}_{n-1}$ and then, $[\partial_{x^\xi}, \mathcal{P}_{D(n-1)}]$ act on functions whose nontrivial dependence is in less than $n-1$ relative distances.

Thus, we can consider (4.6) as equations for $\{\gamma_{n;\,\xi}\}_\xi$ for fixed n, whose right hand side is determined by $\{\gamma_{m;\,\xi}\}_\xi$ for $m = 2, \dots, n-1$. We shall call the maps $\gamma_{n;\,\xi}$ renormalization "cocycles" without meaning of closedness with respect to some differential.

There are important additional restrictions to the solutions of Eqs. (4.6), which are interesting for us. These are the conditions

$$[x^\eta, \gamma_{n;\,\xi}] = 0\,, \tag{4.7}$$

$$\mathrm{sc.d.}\, \gamma_{n;\,\xi}\, f \leqslant 1 + \mathrm{sc.d.}\, f \tag{4.8}$$

($\xi, \eta = 1, \dots, D(n-1)$, $f \in \mathcal{C}_t^\infty(\widetilde{F}_{n-1})$), which have to satisfy all renormalization cocycles defined by (4.4) (the first ones are due to Eqs. (4.5) and conditions ($r3$) and ($p3$), and the second ones are due to ($r2$) and ($p2$)).

Theorem 4.1. ([10]) *Let $n > 2$ and we have a system of primary renormalization maps $\mathcal{P}_D$, $\mathcal{P}_{2D}$, $\dots$, $\mathcal{P}_{D(n-2)}$ (which therefore determine renormalization maps R_2, R_3, $\dots$, R_{n-1}). Then every solution $\{\gamma_{n;\,\xi}\}_\xi$ of Eqs. (4.6) (with fixed n) that satisfies conditions (4.7) and (4.8) corresponds by Eq. (4.4) to some primary renormalization map $\mathcal{P}_{D(n-1)}$ (and hence, to some R_n).*

5 Reduction to de Rham cohomologies

The linear maps $\gamma_{n;\,\xi}$ (4.5) that determine the renormalization cocycles $\omega_{n;\,\xi}$ take values that are distributions supported at $\mathbf{0} \in \mathbb{R}^{D(n-1)}$. Let us expand them in delta functions and their derivatives:

$$\gamma_{n;\,\xi} = \sum_{\mathrm{r}\,\in\,\mathbb{N}_0^N} \frac{1}{\mathrm{r}!}\, \delta^{(\mathrm{r})}(\mathbf{x})\, \Gamma_{n;\,\xi;\,\mathrm{r}}\,, \qquad \Gamma_{n;\,\xi;\,\mathrm{r}} : \mathcal{C}_t^\infty(\widetilde{F}_{n-1}) \to \mathbb{C} \tag{5.1}$$

($N = D(n-1)$, $\delta^{(\mathrm{r})}(\mathbf{x}) := \partial_{\mathbf{x}}^{\mathrm{r}} \delta(\mathbf{x})$, $\partial_{\mathbf{x}}^{\mathrm{r}} := \prod_\xi \partial_{x^\xi}^{r_\xi}$). Note that the sum in (5.1) becomes finite only after applying both sides to a function. Thus we get a characterization of $\gamma_{n;\,\xi}$ by an infinite set of linear functionals $\Gamma_{n;\,\xi;\,\mathrm{r}}$ on $\mathcal{C}_t^\infty(\widetilde{F}_{n-1})$. In fact, we shall show now that the whole information about $\gamma_{n;\,\xi}$ is contained in the leading term $\Gamma_{n;\,\xi;\,0}$ of the expansion (5.1). This follows if we take into account the relation $[x^\eta, \gamma_{n;\,\xi}] = 0$. Combining the latter identity with (5.1) gives a recursive relation $-(r_\eta + 1)\, \Gamma_{n;\,\xi;\,\mathrm{r}+\mathrm{e}_\eta} = \Gamma_{n;\,\xi;\,\mathrm{r}} \circ x^\eta$, which then implies that

$$\Gamma_{n;\,\xi;\,\mathrm{r}} = (-1)^{|\mathrm{r}|}\, \Gamma_{n;\,\xi;\,0} \circ \mathbf{x}^{\mathrm{r}} \tag{5.2}$$

($\mathbf{x}^{\mathrm{r}} := \prod_\xi (x^\xi)^{r_\xi}$). We set

$$\Gamma_{n;\,\xi} := \Gamma_{n;\,\xi;\,0} \tag{5.3}$$

for $\xi = 1, \ldots, D(n-1)$ and organize them as a 1–form

$$\mathbf{\Gamma}_n := \sum_{\xi=1}^{D(n-1)} \Gamma_{n;\xi} \; dx^{\xi} \tag{5.4}$$

with coefficients in the dual differential module $\mathcal{C}_t^{\infty}(\widetilde{F}_{n-1})'$, i.e.,

$$\mathbf{\Gamma}_n \in \Omega^1\Big(\mathcal{C}_t^{\infty}(\widetilde{F}_{n-1})'\Big).$$

In more details, we fix the above notions in the following definitions.

Definition 5.1. Let us introduce the associative algebra $\mathfrak{D}_N$ of all linear partial differential operators over $\mathbb{R}^N$ with polynomial coefficients[2]. A $\mathbb{Z}$–**filtered** $\mathfrak{D}_N$–module is a module $\mathscr{F}$ of $\mathfrak{D}_N$, which is endowed with an increasing filtration $\mathscr{F} = \bigcup\limits_{L \in \mathbb{Z}} \mathscr{F}_{(L)}$, $\mathscr{F}_{(L)} \subseteq \mathscr{F}_{(L+1)}$, such that for every $A \in \mathfrak{D}_N$ and $u \in \mathscr{F}$ we have $\mathrm{sc.d.}\, Au \leqslant \mathrm{sc.d.}\, A + \mathrm{sc.d.}\, u$, where $\mathrm{sc.d.}\, u := \min\{L : u \in \mathscr{F}_{(L)}\}$ and the scaling degree of a differential operator is defined by: $\mathrm{sc.d.}\left(\sum\limits_{\mathrm{r} \in \mathbb{N}_0^N} f_{\mathrm{r}}(\mathrm{x})\, \partial^{\mathrm{r}}\right) = \max\limits_{\mathrm{r} \in \mathbb{N}_0^N} \{|\mathrm{r}| + \mathrm{sc.d.}\, f_{\mathrm{r}}\}$.

The vector space $\mathcal{C}_t^{\infty}(\widetilde{F}_n)$ is an example of a $\mathbb{Z}$–filtered $\mathfrak{D}_{Dn}$ if we use the integer–valued scaling degree on it (i.e., the the smallest integer bigger than or equal to the real scaling degree). We shall denote this differential module by

$$\mathscr{E}_n^{\infty} := \mathcal{C}_t^{\infty}(\widetilde{F}_n). \tag{5.5}$$

If we do perturbation theory starting from massless QFT then we have to apply our renormalization cocycles to a smaller differential module, the algebra of all rational functions in $(\mathrm{x}_1, \ldots, \mathrm{x}_n) \in \mathbb{R}^{Dn}$ with denominators that are products of all quadrics x_k^2 and x_{jk}^2, where $\mathrm{x}_k^2 := \sum\limits_{\mu=1}^{D} (x_k^{\mu})^2$ and $\mathrm{x}_{jk} := \mathrm{x}_j - \mathrm{x}_k$. We denote the latter $\mathfrak{D}_{Dn}$–module by

$$\mathscr{E}_n^0 := \mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_n]\left[\Big(\prod_{k=1}^{n-1} \mathrm{x}_k^2\Big)^{-1} \Big(\prod_{1 \leqslant j < k \leqslant n-1} \mathrm{x}_{jk}^2\Big)^{-1}\right] \subset \mathscr{E}_n^{\infty}. \tag{5.6}$$

Thus, $\Gamma_{n;\xi}$ belong to the dual module of $\mathscr{E}_{n-1}^{\infty}$ generally defined by:

Definition 5.2. For a $\mathfrak{D}_N$–module $\mathscr{F}$ the **dual** module is the algebraic dual space $\mathscr{F}'$ endowed with the dual action of $\mathfrak{D}_N$: for $\Phi \in \mathscr{F}'$ and $\xi = 1, \ldots, N$

[2] or instead of that, we can consider everywhere smooth coefficients, but for us polynomials will be enough.

the dual actions of x^ξ and ∂_{x^ξ} are $x^\xi(\Phi) := \Phi \circ x^\xi$ and $\partial_{x^\xi}(\Phi) := -\Phi \circ \partial_{x^\xi}$, respectively.

Let us point out that the dual differential module $\mathscr{F}'$ of a $\mathbb{Z}$–filtered $\mathfrak{D}_N$–module $\mathscr{F}$ is not naturally $\mathbb{Z}$–graded. But it has a differential submodule that is $\mathbb{Z}$–graded. A simple computation shows that

$$\mathscr{F}^\bullet := \bigcup_{M \in \mathbb{Z}} \mathscr{F}^\perp_{(M)} \ \ (\subseteq \mathscr{F}'), \quad \mathscr{F}^\perp_{(M)} := \Big\{ \Phi \in \mathscr{F}' : \Phi \big|_{\mathscr{F}_{(M)}} = 0 \Big\}$$

is a $\mathfrak{D}_N$–submodule of $\mathscr{F}'$ and it becomes $\mathbb{Z}$–filtered with an increasing $\mathbb{Z}$–filtration if we set

$$\mathscr{F}^\bullet_{(M)} := \mathscr{F}^\perp_{(-M)} .$$

The linear functionals $\Gamma_{n;\,\xi}$ that characterize the cocycles $\gamma_{n;\,\xi}$ are elements of $(\mathscr{E}^0_{n-1})^\bullet$. We shall show this in a slightly more general situation.

Let $\mathscr{F}$ be a $\mathbb{Z}$–graded $\mathfrak{D}_N$–module and denote

$$\mathcal{R}(\mathscr{F}) := \Big\{ \phi : \phi \text{ linearly maps } \mathscr{F} \text{ to } \mathscr{D}'[0],\ [x^\xi, \phi] = 0 \ (\forall \xi = 1, \dots, N), \\ \exists\, L \in \mathbb{Z} \text{ such that } \mathrm{sc.d.}\, \phi(u) \leqslant L + \mathrm{sc.d.}\, u \ (\forall u \in \mathscr{F}) \Big\} .$$

(so for example, $\gamma_{n;\,\xi} \in \mathcal{R}(\mathscr{E}^0_{n-1})$). Expand $\phi \in \mathcal{R}(\mathscr{F})$ in delta functions and their derivatives:

$$\phi = \sum_{\mathrm{r} \in \mathbb{N}_0^N} \frac{1}{\mathrm{r}!}\, \delta^{(\mathrm{r})}(\mathbf{x})\, \Phi_{\mathrm{r}} , \quad \Phi_{\mathrm{r}} : \mathscr{F} \to \mathbb{R} . \tag{5.7}$$

Then the assignment

$$\phi \ \mapsto \ \Phi_0 \tag{5.8}$$

is injective and ϕ is determined by Φ_0 by the formula

$$\Phi_{\mathrm{r}} = (-1)^{|\mathrm{r}|}\, \Phi_0 \circ \mathbf{x}^{\mathrm{r}} . \tag{5.9}$$

This is proven exactly as above for the case of $\gamma_{n;\,\xi}$. Furthermore, under the assignment (5.8):

$$\text{if} \quad \phi \mapsto \Phi_0 \quad \text{then} \quad [\partial_{x^\xi}, \phi] \mapsto -\Phi_0 \circ \partial_{x^\xi} . \tag{5.10}$$

Under the above definitions it is easy to obtain the following proposition (see [10]).

Proposition 5.1. *The image of $\mathcal{R}(\mathscr{F})$ under the map (5.8) is the vector space $\mathscr{F}^\bullet$.*

Corollary 5.2. *If $\gamma_{n;\,\xi}$ are renormalization cocycles (4.4) then the linear functional $\Gamma_{n;\,\xi}$ (5.3) belong to $(\mathscr{E}_{n-1}^{\infty})^{\bullet}$. Conversely, every set of functionals $\Gamma_{n;\,\xi} \in (\mathscr{E}_{n-1}^{\infty})^{\bullet}$ determine by Eqs. (5.2), (5.3) and (5.1) a set of linear maps $\gamma_{n;\,\xi} : \mathscr{E}_{n-1}^{\infty} \to \mathscr{D}'[\mathbf{0}]$ that satisfy the conditions $[x^{\eta}, \gamma_{n;\,\xi}] = 0$.*

Thus, we see that the cohomology behind Eqs. (4.6) is exactly the de Rham cohomology of the differential module $(\mathscr{E}_{n-1}^{\infty})^{\bullet}$. Let us recall its general definition.

Definition 5.3. The *de Rham complex* for an arbitrary $\mathfrak{D}_N$–module $\mathscr{F}$ is defined as the complex:

$$\{0\} \xrightarrow{d} \Omega^0(\mathscr{F}) \xrightarrow{d} \Omega^1(\mathscr{F}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^N(\mathscr{F}) \xrightarrow{d} \{0\}\,, \qquad (5.11)$$

where

$$\Omega^0(\mathscr{F}) \,\equiv\, \mathscr{F}\,, \quad \Omega^m(\mathscr{F}) \,:=\, \Lambda^m(\mathbb{R}^N) \otimes \mathscr{F}\,,$$

and $\Lambda^m(\mathbb{R}^N)$ stands for the mth antisymmetric power of $\mathbb{R}^N$. Thus, the elements of $\Omega^m(\mathscr{F})$ are represented by sequences $\mathbf{\Theta} = (\Theta_{\xi_1,\dots,\xi_m})$ with coefficients $\Theta_{\xi_1,\dots,\xi_m} \in \mathscr{F}$ for $\xi_1, \dots, \xi_m = 1, \dots, N$, which are antisymmetric, $\Theta_{\xi_1,\dots,\xi_m} = (-1)^{\mathrm{sgn}\,\sigma}\, \Theta_{\xi_{\sigma_1},\dots,\xi_{\sigma_m}}$. The differential of $\Theta_{\xi_1,\dots,\xi_m}$ is

$$(d\mathbf{\Theta})_{\xi_1,\dots,\xi_{m+1}} \,=\, \sum_{\ell\,=\,1}^{m} (-1)^{\ell+1}\, \partial_{x^{\xi_\ell}}\, \Theta_{\xi_1,\dots,\widehat{\xi_\ell},\dots,\xi_{m+1}}\,. \qquad (5.12)$$

Denote by $H^m(\mathscr{F})$, for $m = 0, \dots, N$, the cohomology group of the complex (5.11):

$$H^m(\mathscr{F}) \,:=\, \mathcal{Z}^m(\mathscr{F}) \Big/ \mathcal{B}^m(\mathscr{F})\,,$$

$$\mathcal{Z}^m(\mathscr{F}) \,:=\, Ker\ d\big|_{\Omega^m(\mathscr{F})}\,, \quad \mathcal{B}^m(\mathscr{F}) \,:=\, d(\Omega^{m-1}(\mathscr{F}))\,.$$

The following result is one of the main ingredients in the proof of Theorem 4.1.

Theorem 5.3. ([10]) *There exist the following isomorphisms for $n \geqslant 2$:*

$$H^1\Big((\mathscr{E}_n^{\infty})^{\bullet}\Big)' \,\cong\, H^{Dn-1}(\mathscr{E}_n^{\infty}) \,\cong\, H^{Dn-1}(\widetilde{F}_n)\,.$$

6 Cohomological equations. Outlook

We conclude the paper with a discussion how the cohomological equations (4.6) can be further reduced to equations for the 1–forms (5.4):

$$\mathbf{\Gamma}_n \,\in\, \Omega^1\Big((\mathscr{E}_{n-1}^{\infty})^{\bullet}\Big)$$

that characterize the renormalization cocycles. The resulting equations are in general of the form:

$$\begin{aligned} d\mathbf{\Gamma}_2 &= 0, \\ d\mathbf{\Gamma}_n &= \mathcal{F}_n[\mathbf{\Gamma}_1, \dots, \mathbf{\Gamma}_{n-1}] \qquad (n > 2)\,. \end{aligned} \tag{6.1}$$

In order to derive the right hand side of (6.1) in a simple explicit form we further restrict the renormalization map R_n (for every $n = 2, 3, \dots$) to the subspace $\mathscr{E}^0_{n-1}$ (see (5.6)). The latter consists of all *finite sums of products* of type

$$u \,=\, \Big(\prod_{1 \leqslant j < k \leqslant n-1} G_{j,k}(\mathrm{x}_j - \mathrm{x}_k) \Big) \Big(\prod_{\ell = 1}^{n-1} G_{\ell,n}(\mathrm{x}_\ell) \Big) \tag{6.2}$$

where $G_{j,k}, G_{\ell,n} \in \mathbb{C}[\mathrm{x}][1/\mathrm{x}^2] \subset \mathcal{C}^\infty_t(\mathbb{R}^D \backslash \{\mathbf{0}\})$ (recall that x_k have now meaning of relative distances). Thus, we shall look for $\mathbf{\Gamma}_n$ as an elements of the $\mathfrak{D}_{D(n-1)}$–module $\Omega^1\big((\mathscr{E}^0_n)^\bullet\big)$. Let us introduce, for every proper subset $S \subset \mathbb{n} := \{1, \dots, n\}$, maps $\gamma_{S;\,\xi}$ $(\xi = 1, \dots, D(n-1))$ as acting on u (6.2) in the following way:

$$\gamma_{S;\,\xi}(u) \,:=\, u^{\mathrm{c}}_S \cdot \gamma_{|S|;\,\xi}(u_S)$$

where u_S is the part of the two products in Eq. (6.2), which consists of all $G_{j',k'}$ with $j' < k'$ and $j', k' \in S$, and u^{c}_S stands for the remaining part of the products. The maps $\gamma_{S;\,\xi}$ are considered as linear maps between the spaces:

$$\gamma_{S;\,\xi} : \mathscr{E}^0_{n-1} \to \mathscr{E}^0_{\mathbb{n}/S} \otimes \mathscr{D}'[\mathbf{0}]\,,$$

where $\mathscr{D}'[\mathbf{0}]$ is the space of distributions supported at $\mathbf{0} \in \mathbb{R}^{D(m-1)}$ (with respect to the variables related to $S \subset \mathbb{n}$), and we have also set

$$\mathbb{n}/S \,:=\, (\mathbb{n} \backslash S) \cup \{\max\, S\} \qquad (|\mathbb{n}/S| = \, n - |S| + 1)\,,$$

and $\mathscr{E}^0_S$ (for any $S \subseteq \mathbb{n}$) stands for the space of finite sums of products $\prod\limits_{\substack{j,\,k\,\in\,S \\ j\,<\,k}} G_{j,k}$. Under these conventions one can derive the equalities:

$$[\partial_{x^\mu_k}, \widehat{R}_n] \,=\, \sum_{\substack{S \,\subset\, \mathbb{n} \\ |S| \,\geqslant\, 2}} \Big(\widehat{R}_{\mathbb{n}/S} \otimes id_{\mathscr{D}'[0]} \Big) \circ \gamma_{S;\,k,\,\mu}\,. \tag{6.3}$$

Equations (6.3) are derived under some additional conditions on the renormalization maps R_n (which break the permutation symmetry but it can be always restored by a symmetrization at the end).

Let $\Gamma_{S;\,\xi}$ correspond to $\gamma_{S;\,\xi}$ under the assignment (5.3) and the decomposition (5.1). These maps $\Gamma_{S;\,\xi}$ are now linear maps

$$\Gamma_{S;\,\xi} : \mathscr{E}_{n-1}^{0} \to \mathscr{E}_{\mathbb{n}/S}^{0}$$

and we collect them into a 1–form $\mathbf{\Gamma}_S$ similarly to $\mathbf{\Gamma}_n$. So, $\mathbf{\Gamma}_S$ is basically $\mathbf{\Gamma}_{|S|}$ but extended to $\mathscr{E}_{n-1}^{0}$ as a collection of maps commuting with the multiplication by functions that do not have singularities with respect to $\mathrm{x}_j - \mathrm{x}_k$ for all $j, s \in S$.

Finally, under all the above considerations one can derive the following counterpart of the cohomological equations (4.6):

$$\begin{aligned} d\mathbf{\Gamma}_2 &= 0\,, \\ d\mathbf{\Gamma}_n &= - \sum_{\substack{S \subset \mathbb{n} \\ |S| \geqslant 2}} \mathbf{\Gamma}_{\mathbb{n}/S} \wedge \mathbf{\Gamma}_S \qquad (n > 2)\,. \end{aligned} \tag{6.4}$$

We intend to study the cohomological equations (6.4) and their solutions in the future. They can be also considered on the differential submodule $\mathscr{E}_{n-1}^{0}$. We point out on $\mathscr{E}_{n}^{0}$ Theorem 4.1 is no longer valid since we have

$$H^1\Big((\mathscr{E}_n^0)^{\bullet}\Big)' \cong H^{Dn-1}(\mathscr{E}_n^0) \neq \{0\}\,.$$

But on the other hand the space $\mathscr{E}_n^{\infty}$ is too large to work "algebraically" on it. Thus we arrive to the question if there exists some differential extension

$$\mathscr{E}_n^0 \subset \mathscr{E}_n$$

of $\mathscr{E}_{n-1}^{0}$, which is minimal in some sense and has trivial cohomologies. In fact, one possible strategy for solving Eqs. (6.4) would be to find a homotopy operator

$$K_n \,:\, \Omega^{Dn-1}(\mathscr{E}_n) \to \Omega^{Dn-2}(\mathscr{E}_n)\,, \quad K_n \circ d + d \circ K_n \,=\, \mathrm{id}\,,$$

and then a solution of (6.1) for $n > 2$ will be

$$\mathbf{\Gamma}_n \,=\, \mathcal{F}_n \circ K_{n-1}\,.$$

Let us note that for the case "$D = 1$" such a differential extension is given by a *polylogarithmic extension* of the ring of rational functions $\mathscr{E}_n^0$. These extensions are studied in [1].

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