New Prospects in Unified Theories: From Finiteness to Fuzzy Extra Dimensions*

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Abstract

Finite Unified Theories (FUTs) are N=1 supersymmetric Grand Unified Theories, which can be made all-loop finite, both in the dimensionless (gauge and Yukawa couplings) and dimensionful (soft supersymmetry breaking terms) sectors. This remarkable property, based on the reduction of couplings at the quantum level, provides a drastic reduction in the number of free parameters. By confronting the predictions of $SU(5)$ FUTs with the top and bottom quark masses we are able to discriminate among different models, including further low-energy phenomenology constraints, we derive predictions for the lightest Higgs boson mass and the sparticle spectrum. Next we consider gauge theories defined in higher dimensions, where the extra dimensions form a fuzzy space (a finite matrix manifold). We reinterpret these gauge theories as four-dimensional theories with Kaluza-Klein modes. We then perform a generalized à la Forgacs-Manton dimensional reduction. We emphasize some striking features emerging such as (i) the appearance of non-abelian gauge theories in four dimensions starting from an abelian gauge theory in higher dimensions, (ii) the fact that the spontaneous symmetry breaking of the theory takes place entirely in the extra dimensions and (iii) the renormalizability of the theory both in higher as well as in four dimensions. Then reversing the above approach we present a renormalizable four dimensional $SU(N)$ gauge theory with a suitable multiplet of scalar fields, which via spontaneous symmetry breaking dynamically develops extra dimensions in

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the form of a fuzzy sphere S^2_N. We explicitly find the tower of massive Kaluza-Klein modes consistent with an interpretation as gauge theory on $M^4 \times S^2$, the scalars being interpreted as gauge fields on S^2. Depending on the parameters of the model the low-energy gauge group can be $SU(n)$, or broken further to $SU(n_1) \times SU(n_2) \times U(1)$. Therefore the second picture justifies the first one in a renormalizable framework but in addition has the potential to reveal new aspects of the theory.

1. Introduction

The theoretical efforts to establish a deeper understanding of Nature have led to very interesting frameworks such as String theories and Non-commutative Geometry both of which aim to describe physics at the Planck scale. Looking for the origin of the idea that coordinates might not commute we might have to go back to the days of Heisenberg. In the recent years the birth of such speculations can be found in refs. [1, 2]. In the spirit of Non-commutative Geometry also particle models with non-commutative gauge theory were explored [3] (see also [4]), [5, 6, 7, 8, 9]. On the other hand the present intensive research has been triggered by the natural realization of non-commutativity of space in the string theory context of D-branes in the presence of a constant background antisymmetric field [10]. After the work of Seiberg and Witten [11], where a map (SW map) between non-commutative and commutative gauge theories has been described, there has been a lot of activity also in the construction of non-commutative phenomenological Lagrangians, for example various non-commutative standard model like Lagrangians have been proposed [12, 13][1]. In particular in ref. [13], following the SW map methods developed in refs. [16, 17, 18, 19], a non-commutative standard model with $SU(3) \times SU(2) \times U(1)$ gauge group has been presented. These non-commutative models represent interesting generalizations of the SM and hint at possible new physics. However they do not address the usual problem of the SM, the presence of a plethora of free parameters mostly related to the ad hoc introduction of the Higgs and Yukawa sectors in the theory. At this stage it is worth recalling that various schemes, with the Coset Space Dimensional Reduction (CSDR) [20, 21, 22, 23, 24] being pioneer, were suggesting that a unification of the gauge and Higgs sectors can be achieved in higher dimensions. Moreover the addition of fermions in the higher-dimensional gauge theory leads naturally after CSDR to Yukawa couplings in four dimensions. In the successes of the CSDR scheme certainly should be added the possibility to obtain chiral theories in four dimensions [25, 26, 27, 28, 29, 30, 31, 32] as well as softly broken supersymmetric or non-supersymmetric theories starting from a supersymmetric gauge theory defined in higher dimensions [33].

[1]These SM actions are mainly considered as effective actions because they are not renormalizable. The effective action interpretation is consistent with the SM in [13] being anomaly free [14]. Non-commutative phenomenology has been discussed in [15].

The original plan of this paper was to present an overview covering the following subjects:

a) Quantum Reduction of Couplings and Finite Unified Theories
b) Classical Reduction of Couplings and Coset Space Dimensional Reduction
c) Renormalizable Unified Theories from Fuzzy Higher Dimensions [34, 35, 36, 37]

The aim was to present an unified description of our current attempts to reduce the free parameters of the Standard Model by using Finite Unification and extra dimensions. However we will cover only the first and the third subjects given the fact that there exists extensive reviews covering a major part of the second one [21, 22]. These two topics represent different attempts at reduction of couplings, on one hand the Finite Unified Theories showing promising models with good phenomenology, on the other hand, the Unified Theories from Fuzzy Higher Dimensions combining dimensional reduction and reduction of couplings in a renormalizable theory.

2. Reduction of Couplings and Finiteness in $N = 1$ SUSY Gauge Theories

Finite Unified Theories are $N = 1$ supersymmetric Grand Unified Theories (GUTs) which can be made finite even to all-loop orders, including the soft supersymmetry breaking sector. The method to construct GUTs with reduced independent parameters [38, 39, 40, 41, 42] consists of searching for renormalization group invariant (RGI) relations holding below the Planck scale, which in turn are preserved down to the GUT scale. Of particular interest is the possibility to find RGI relations among couplings that guarantee finiteness to all-orders in perturbation theory [43, 44, 45, 46, 47, 48]. In order to achieve the latter it is enough to study the uniqueness of the solutions to the one-loop finiteness conditions [43, 44, 45, 46, 47, 48]. The constructed *finite unified* $N = 1$ supersymmetric SU(5) GUTs, using the above tools, predicted correctly from the dimensionless sector (Gauge-Yukawa unification), among others, the top quark mass [49, 50]. The search for RGI relations and finiteness has been extended to the soft supersymmetry breaking sector (SSB) of these theories [51, 52], which involves parameters of dimension one and two. Eventually, the full theories can be made all-loop finite and their predictive power is extended to the Higgs sector and the supersymmetric spectrum (s-spectrum).

Here let us review the main points and ideas concerning the *reduction of couplings* and *finiteness* in $N = 1$ supersymmetric theories. A RGI relation among couplings g_i, $\Phi(g_1, \cdots, g_N) = 0$, has to satisfy the partial differential equation $\mu\, d\Phi/d\mu = \sum_{i=1}^{N} \beta_i\, \partial\Phi/\partial g_i = 0$, where β_i is the β-function of g_i. There exist $(N-1)$ independent Φ's, and finding the complete set of these solutions is equivalent to solve the so-called reduction equations (REs) [39, 40, 41, 42], $\beta_g\,(dg_i/dg) = \beta_i$, $i = 1, \cdots, N$, where g and β_g are the primary coupling and its β-function. Using all the $(N-1)$ Φ's to impose

RGI relations, one can in principle express all the couplings in terms of a single coupling g. The complete reduction, which formally preserves perturbative renormalizability, can be achieved by demanding a power series solution, whose uniqueness can be investigated at the one-loop level.

Finiteness can be understood by considering a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory based on a group G with gauge coupling constant g. The superpotential of the theory is given by

$$W = \frac{1}{2}\, m^{ij}\, \Phi_i\, \Phi_j + \frac{1}{6}\, C^{ijk}\, \Phi_i\, \Phi_j\, \Phi_k \ , \tag{1}$$

where m^{ij} (the mass terms) and C^{ijk} (the Yukawa couplings) are gauge invariant tensors and the matter field Φ_i transforms according to the irreducible representation R_i of the gauge group G.

The one-loop β-function of the gauge coupling g is given by

$$\beta_g^{(1)} = \frac{dg}{dt} = \frac{g^3}{16\pi^2}\,[\sum_i T(R_i) - 3\, C_2(G)\,] \ , \tag{2}$$

where $T(R_i)$ is the Dynkin index of R_i and $C_2(G)$ is the quadratic Casimir of the adjoint representation of the gauge group G. The β-functions of C^{ijk}, by virtue of the non-renormalization theorem, are related to the anomalous dimension matrix γ_i^j of the matter fields Φ_i as:

$$\beta_C^{ijk} = \frac{d}{dt}\, C^{ijk} = C^{ijp} \sum_{n=1} \frac{1}{(16\pi^2)^n}\, \gamma_p^{k(n)} + (k \leftrightarrow i) + (k \leftrightarrow j) \ . \tag{3}$$

At one-loop level γ_i^j is given by

$$\gamma_i^{j(1)} = \frac{1}{2} C_{ipq}\, C^{jpq} - 2\, g^2\, C_2(R_i) \delta_i^j \ , \tag{4}$$

where $C_2(R_i)$ is the quadratic Casimir of the representation R_i, and $C^{ijk} = C^*_{ijk}$.

All the one-loop β-functions of the theory vanish if the β-function of the gauge coupling $\beta_g^{(1)}$, and the anomalous dimensions $\gamma_i^{j(1)}$, vanish, i.e.

$$\sum_i T(R_i) = 3C_2(G)\,, \qquad \frac{1}{2}\, C_{ipq} C^{jpq} = 2\delta_i^j g^2 C_2(R_i) \ . \tag{5}$$

The conditions for finiteness for $N = 1$ field theories with $SU(N)$ gauge symmetry are discussed in [53], and the analysis of the anomaly-free and no-charge renormalization requirements for these theories can be found in [54]. A very interesting result is that the conditions (5) are necessary and sufficient for finiteness at the two-loop level [55, 56].

The one- and two-loop finiteness conditions (5) restrict considerably the possible choices of the irreducible representations R_i for a given group G as well as the Yukawa couplings in the superpotential (1). Note in particular that the finiteness conditions cannot be applied to the supersymmetric standard model (SSM), since the presence of a $U(1)$ gauge group is incompatible with the condition (5), due to $C_2[U(1)] = 0$. This leads to the expectation that finiteness should be attained at the grand unified level only, the SSM being just the corresponding low-energy, effective theory.

The finiteness conditions impose relations between gauge and Yukawa couplings. Therefore, we have to guarantee that such relations leading to a reduction of the couplings hold at any renormalization point. The necessary, but also sufficient, condition for this to happen is to require that such relations are solutions to the reduction equations (REs) to all orders. The all-loop order finiteness theorem of ref.[43, 44, 45, 46] is based on: (a) the structure of the supercurrent in $N = 1$ SYM and on (b) the non-renormalization properties of $N = 1$ chiral anomalies [43, 44, 45, 46]. Alternatively, similar results can be obtained [47, 48, 57] using an analysis of the all-loop NSVZ gauge beta-function [58, 59, 60].

3. Soft supersymmetry breaking and finiteness

The above described method of reducing the dimensionless couplings has been extended [52, 51] to the soft supersymmetry breaking (SSB) dimensionful parameters of $N = 1$ supersymmetric theories. Important results have been made [61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76] concerning the renormalization properties of the SSB parameters, based conceptually and technically on the work of ref. [65]. In this work the powerful supergraph method [68, 69, 70, 71] for studying supersymmetric theories has been applied to the softly broken ones by using the "spurion" external space-time independent superfields [72]. In the latter method a softly broken supersymmetric gauge theory is considered as a supersymmetric one in which the various parameters such as couplings and masses have been promoted to external superfields that acquire "vacuum expectation values". Based on this method the relations among the soft term renormalization and that of an unbroken supersymmetric theory have been derived. In particular the β-functions of the parameters of the softly broken theory are expressed in terms of partial differential operators involving the dimensionless parameters of the unbroken theory. The key point in the strategy of refs. [61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76] in solving the set of coupled differential equations so as to be able to express all parameters in a RGI way, was to transform the partial differential operators involved to total derivative operators [61, 62]. It is indeed possible to do this on the RGI surface which is defined by the solution of the reduction equations. In addition it was found that RGI SSB scalar masses in Gauge-Yukawa unified models satisfy a universal sum rule at one-loop [67]. This result was generalized to two-loops for finite theories [74, 75, 76], and then to all-loops for general Gauge-Yukawa and Finite Unified Theories [63, 64].

In order to obtain a feeling of some of the above results, consider the superpotential given by (1) along with the Lagrangian for SSB terms

$$-\mathcal{L}_{\rm SB} = \frac{1}{6}\, h^{ijk}\, \phi_i\phi_j\phi_k + \frac{1}{2}\, b^{ij}\, \phi_i\phi_j + \frac{1}{2}\,(m^2)^j_i\, \phi^{*\,i}\phi_j + \frac{1}{2}\, M\, \lambda\lambda + {\rm H.c.}, \quad (6)$$

where the ϕ_i are the scalar parts of the chiral superfields Φ_i , λ are the gauginos and M their unified mass. Since only finite theories are considered here, it is assumed that the gauge group is a simple group and the one-loop β-function of the gauge coupling g vanishes. It is also assumed that the reduction equations admit power series solutions of the form

$$C^{ijk} = g \sum_{n=0} \rho^{ijk}_{(n)} g^{2n} \ . \quad (7)$$

According to the finiteness theorem [43, 44, 45, 46], the theory is then finite to all-orders in perturbation theory, if, among others, the one-loop anomalous dimensions $\gamma_i^{j(1)}$ vanish. The one- and two-loop finiteness for h^{ijk} can be achieved by [77]

$$h^{ijk} = -MC^{ijk} + \cdots = -M\rho^{ijk}_{(0)}\, g + O(g^5) \ . \quad (8)$$

An additional constraint in the SSB sector up to two-loops [74, 75, 76], concerns the soft scalar masses as follows

$$\frac{(\ m_i^2 + m_j^2 + m_k^2\ )}{MM^\dagger} = 1 + \frac{g^2}{16\pi^2}\, \Delta^{(2)} + O(g^4) \quad (9)$$

for i, j, k with $\rho^{ijk}_{(0)} \neq 0$, where $\Delta^{(2)}$ is the two-loop correction

$$\Delta^{(2)} = -2\sum_l [(m_l^2/MM^\dagger) - (1/3)]\ T(R_l), \quad (10)$$

which vanishes for the universal choice [77], i.e. when all the soft scalar masses are the same at the unification point.

If we know higher-loop β-functions explicitly, we can follow the same procedure and find higher-loop RGI relations among SSB terms. However, the β-functions of the soft scalar masses are explicitly known only up to two loops. In order to obtain higher-loop results, we need something else instead of knowledge of explicit β-functions, e.g. some relations among β-functions.

By means of the spurion technique [68, 69, 70, 71, 72] it is possible to derive the following all-loop relations among SSB β-functions, [61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76]

$$\beta_M = 2\mathcal{O}\left(\frac{\beta_g}{g}\right) , \tag{11}$$

$$\beta_h^{ijk} = \gamma^i{}_l h^{ljk} + \gamma^j{}_l h^{ilk} + \gamma^k{}_l h^{ijl} - 2\gamma_1^i{}_l C^{ljk} - 2\gamma_1^j{}_l C^{ilk} - 2\gamma_1^k{}_l C^{ijl} , \tag{12}$$

$$(\beta_{m^2})^i{}_j = \left[\Delta + X\frac{\partial}{\partial g}\right]\gamma^i{}_j , \tag{13}$$

$$\mathcal{O} = \left(Mg^2\frac{\partial}{\partial g^2} - h^{lmn}\frac{\partial}{\partial C^{lmn}}\right) , \tag{14}$$

$$\Delta = 2\mathcal{O}\mathcal{O}^* + 2|M|^2 g^2\frac{\partial}{\partial g^2} + \tilde{C}_{lmn}\frac{\partial}{\partial C_{lmn}} + \tilde{C}^{lmn}\frac{\partial}{\partial C^{lmn}} , \tag{15}$$

where $(\gamma_1)^i{}_j = \mathcal{O}\gamma^i{}_j$, $C_{lmn} = (C^{lmn})^*$, and

$$\tilde{C}^{ijk} = (m^2)^i{}_l C^{ljk} + (m^2)^j{}_l C^{ilk} + (m^2)^k{}_l C^{ijl} . \tag{16}$$

It was also found [73] that the relation

$$h^{ijk} = -M(C^{ijk})' \equiv -M\frac{dC^{ijk}(g)}{d\ln g} , \tag{17}$$

among couplings is all-loop RGI. Furthermore, using the all-loop gauge β-function of Novikov *et al.* [58, 59, 60] given by

$$\beta_g^{\rm NSVZ} = \frac{g^3}{16\pi^2}\left[\frac{\sum_l T(R_l)(1-\gamma_l/2) - 3C(G)}{1 - g^2C(G)/8\pi^2}\right] , \tag{18}$$

it was found the all-loop RGI sum rule [63, 64],

$$\begin{aligned} m_i^2 + m_j^2 + m_k^2 = |M|^2\{ &\frac{1}{1-g^2C(G)/(8\pi^2)}\frac{d\ln C^{ijk}}{d\ln g} \\ &+\frac{1}{2}\frac{d^2\ln C^{ijk}}{d(\ln g)^2}\} + \sum_l \frac{m_l^2 T(R_l)}{C(G) - 8\pi^2/g^2}\frac{d\ln C^{ijk}}{d\ln g} . \end{aligned} \tag{19}$$

In addition the exact-β-function for m^2 in the NSVZ scheme has been obtained [63, 64] for the first time and is given by

$$\begin{aligned} \beta_{m_i^2}^{\rm NSVZ} = \Bigg[&|M|^2\{ \frac{1}{1-g^2C(G)/(8\pi^2)}\frac{d}{d\ln g} + \frac{1}{2}\frac{d^2}{d(\ln g)^2} \} \\ &+ \sum_l \frac{m_l^2 T(R_l)}{C(G) - 8\pi^2/g^2}\frac{d}{d\ln g} \Bigg] \gamma_i^{\rm NSVZ} . \end{aligned} \tag{20}$$

4. Finite Unified Theories

The first one- and two-loop $SU(5)$ finite model was presented in [78]. Here we will examine Finite Unified theories with $SU(5)$ gauge group, where the reduction of couplings has been applied to the third generation of quarks and leptons. An extension to three families, and the generation of quark mixing angles and masses in Finite Unified Theories has been addressed in [79], where several examples are given. These extensions are not considered here. Realistic Finite Unified Theories based on product gauge groups, where the finiteness implies three generations of matter, have also been studied [80].

A predictive Gauge-Yukawa unified $SU(5)$ model which is finite to all orders, in addition to the requirements mentioned already, should also have the following properties:

1. One-loop anomalous dimensions are diagonal, i.e., $\gamma_i^{(1)\,j} \propto \delta_i^j$.
2. Three fermion generations, in the irreducible representations $\overline{\mathbf{5}}_i, \mathbf{10}_i$ $(i = 1, 2, 3)$, which obviously should not couple to the adjoint $\mathbf{24}$.
3. The two Higgs doublets of the MSSM should mostly be made out of a pair of Higgs quintet and anti-quintet, which couple to the third generation.

In the following we discuss two versions of the all-order finite model. The model of ref. [49, 50], which will be labeled **A**, and a slight variation of this model (labeled **B**), which can also be obtained from the class of the models suggested in ref. [61, 62] with a modification to suppress non-diagonal anomalous dimensions.

The superpotential which describes the two models takes the form [49, 50, 74, 75, 76]

$$W = \sum_{i=1}^{3} \Big[\, \frac{1}{2} g_i^u \, \mathbf{10}_i \mathbf{10}_i H_i + g_i^d \, \mathbf{10}_i \overline{\mathbf{5}}_i \, \overline{H}_i \, \Big] + g_{23}^u \, \mathbf{10}_2 \mathbf{10}_3 H_4$$
$$+ g_{23}^d \, \mathbf{10}_2 \overline{\mathbf{5}}_3 \, \overline{H}_4 + g_{32}^d \, \mathbf{10}_3 \overline{\mathbf{5}}_2 \, \overline{H}_4 + \sum_{a=1}^{4} g_a^f \, H_a \, \mathbf{24} \, \overline{H}_a + \frac{g^\lambda}{3} \, (\mathbf{24})^3 \; , \quad (21)$$

where H_a and $\overline{H}_a$ $(a = 1, \ldots, 4)$ stand for the Higgs quintets and anti-quintets.

The non-degenerate and isolated solutions to $\gamma_i^{(1)} = 0$ for the models

$\{\mathbf{A}\ ,\ \mathbf{B}\}$ are:

$$
\begin{aligned}
&(g_1^u)^2 = \{\frac{8}{5}, \frac{8}{5}\}g^2\ ,\ (g_1^d)^2 = \{\frac{6}{5}, \frac{6}{5}\}g^2\ , \\
&(g_2^u)^2 = (g_3^u)^2 = \{\frac{8}{5}, \frac{4}{5}\}g^2\ ,\ (g_2^d)^2 = (g_3^d)^2 = \{\frac{6}{5}, \frac{3}{5}\}g^2\ , \\
&(g_{23}^u)^2 = \{0, \frac{4}{5}\}g^2\ ,\ (g_{23}^d)^2 = (g_{32}^d)^2 = \{0, \frac{3}{5}\}g^2\ ,\ (g^\lambda)^2 = \frac{15}{7}g^2\ , \\
&(g_2^f)^2 = (g_3^f)^2 = \{0, \frac{1}{2}\}g^2\ ,\ (g_1^f)^2 = 0\ ,\ (g_4^f)^2 = \{1, 0\}g^2\ .
\end{aligned}
\tag{22}
$$

According to the theorem of ref. [43, 44, 45, 46] these models are finite to all orders. After the reduction of couplings the symmetry of W is enhanced [49, 74, 75].

The main difference of the models $\mathbf{A}$ and $\mathbf{B}$ is that two pairs of Higgs quintets and anti-quintets couple to the $\mathbf{24}$ for $\mathbf{B}$ so that it is not necessary to mix them with H_4 and $\overline{H}_4$ in order to achieve the triplet-doublet splitting after the symmetry breaking of $SU(5)$. Therefore, the solutions of eq.(5) for the Yukawa couplings are different, as can be seen from eq.(22), which reflects in the phenomenology, as we will see in the next section.

In the dimensionful sector, the sum rule gives us the following boundary conditions at the GUT scale [74, 75]:

$$
m_{H_u}^2 + 2m_{\mathbf{10}}^2 = m_{H_d}^2 + m_{\overline{\mathbf{5}}}^2 + m_{\mathbf{10}}^2 = M^2 \quad \text{for} \quad \mathbf{A}\ ; \tag{23}
$$

$$
\begin{aligned}
&m_{H_u}^2 + 2m_{\mathbf{10}}^2 = M^2\ ,\ m_{H_d}^2 - 2m_{\mathbf{10}}^2 = -\frac{M^2}{3}\ , \\
&m_{\overline{\mathbf{5}}}^2 + 3m_{\mathbf{10}}^2 = \frac{4M^2}{3} \quad \text{for} \quad \mathbf{B},
\end{aligned}
\tag{24}
$$

where we use as free parameters $m_{\overline{\mathbf{5}}} \equiv m_{\overline{\mathbf{5}}_3}$ and $m_{\mathbf{10}} \equiv m_{\mathbf{10}_3}$ for the model $\mathbf{A}$, and $m_{\mathbf{10}} \equiv m_{\mathbf{10}_3}$ for $\mathbf{B}$, in addition to M.

5. Predictions of low energy parameters

Since the gauge symmetry is spontaneously broken below $M_{\rm GUT}$, the finiteness conditions do not restrict the renormalization properties at low energies, and all it remains are boundary conditions on the gauge and Yukawa couplings (22), the $h = -MC$ relation, and the soft scalar-mass sum rule (9) at $M_{\rm GUT}$, as applied in the two models. Thus we examine the evolution of these parameters according to their RGEs up to two-loops for dimensionless parameters and at one-loop for dimensionful ones with the relevant boundary conditions. Below $M_{\rm GUT}$ their evolution is assumed to be governed by the MSSM. We further assume a unique supersymmetry breaking scale M_s (which we define as the geometric mean of the stop masses) and therefore below that scale the effective theory is just the SM.

We now present the comparison of the predictions of the four models with the experimental data, see ref. [81] for more details, starting with the heavy quark masses. In fig.1 we show the **FUTA** and **FUTB** predictions for $M_{\rm top}$ and $m_{\rm bot}(M_Z)$ as a function of the unified gaugino mass M, for the two cases $\mu < 0$ and $\mu > 0$. In the value of the bottom mass $m_{\rm bot}$, we have included the corrections coming from bottom squark-gluino loops and top squark-chargino loops [82]. We give the predictions for the running bottom quark mass evaluated at M_Z, $m_{\rm bot}(M_Z) = 2.825 \pm 0.1$ [83], to avoid the large QCD uncertainties inherent for the pole mass. The value of $m_{\rm bot}$ depends strongly on the sign of μ due to the above mentioned radiative corrections. For both models **A** and **B** the values for $\mu > 0$ are above the central experimental value, with $m_{\rm bot}(M_Z) \sim 4.0 - 5.0$ GeV. For $\mu < 0$, on the other hand, model **B** has overlap with the experimental allowed values, $m_{\rm bot}(M_Z) \sim 2.5 - 2.8$ GeV, whereas for model **A**, $m_{\rm bot}(M_Z) \sim 1.5 - 2.6$ GeV, there is only a small region of allowed parameter space at two sigma level, and only for large values of M. This clearly selects the negative sign of μ.

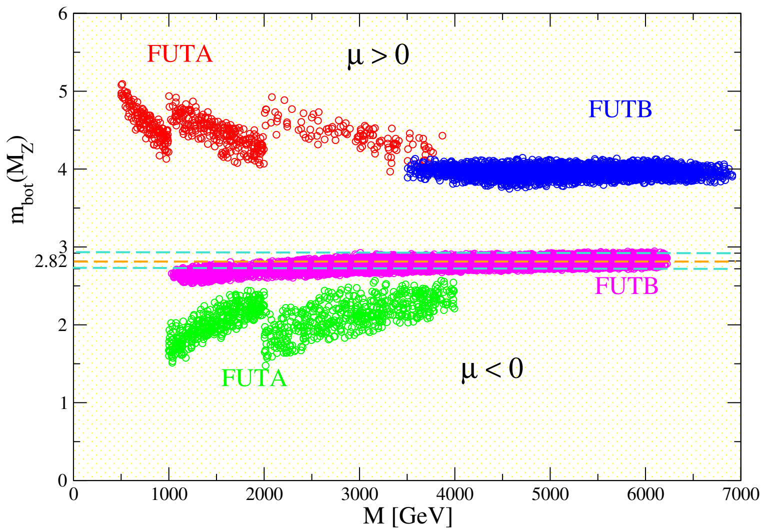
The predictions for the top quark mass $M_{\rm top}$ are ~ 183 and ~ 172 GeV in the models **A** and **B** respectively, as shown in the lower plot of fig. 1. Comparing these predictions with the most recent experimental value $M_{\rm top}^{exp} = (170.9 \pm 1.8)$ GeV [84], and recalling that the theoretical values for $M_{\rm top}$ may suffer from a correction of $\sim 4\%$ [76], we see that clearly model **B** is singled out. In addition the value of $\tan\beta$ is found to be $\tan\beta \sim 54$ and ~ 48 for models **A** and **B**, respectively. Thus from the comparison of the predictions of the two models with experimental data only **FUTB** with $\mu < 0$ survives.

We now analyze the impact of further low-energy observables on the model **FUTB** with $\mu < 0$. In the case where all the soft scalar masses are universal at the unification scale, there is no region of M below $\mathcal{O}$(few TeV) in which $m_{\tilde\tau} > m_{\chi^0}$ is satisfied (where $m_{\tilde\tau}$ is the lightest $\tilde\tau$ mass, and m_{χ^0} the lightest neutralino mass). But once the universality condition is relaxed this problem can be solved naturally, thanks to the sum rule (9). Using this equation and imposing the conditions of (a) successful radiative electroweak symmetry breaking, (b) $m^2_{\tilde\tau} > 0$ and (c) $m_{\tilde\tau} > m_{\chi^0}$, a comfortable parameter space is found for **FUTB** with $\mu < 0$ (and also for **FUTA** and both signs of μ).

As additional constraints we consider the following observables: the rare b decays BR($b \to s\gamma$) and BR($B_s \to \mu^+\mu^-$), the lightest Higgs boson mass as well as the density of cold dark matter in the Universe, assuming it consists mainly of neutralinos. More details and a complete set of references can be found in ref. [81].

Concerning the lightest Higgs boson mass, M_h, the SM bound of 114.4 GeV [85, 86] can be used. For the prediction we use the code `FeynHiggs` [87, 88, 89, 90].

The lightest supersymmetric particle (LSP) is an excellent candidate for cold dark matter (CDM) [91, 92], with a density that falls naturally within



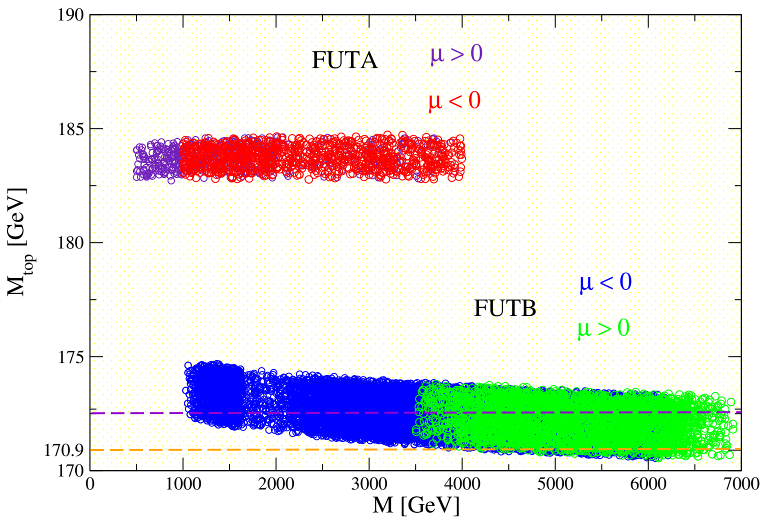


Figure 1: The bottom quark mass at the Z boson scale (upper) and top quark pole mass (lower plot) are shown as function of M for both models.

the range

$$0.094 < \Omega_{\rm CDM} h^2 < 0.129 \tag{25}$$

favored by a joint analysis of WMAP and other astrophysical and cosmological data [93, 94]. Assuming that the cold dark matter is composed predominantly of LSPs, the determination of $\Omega_{\rm CDM} h^2$ imposes very strong constraints on the MSSM parameter space, and we find that no FUT model points fulfill the strict bound of (25). Therefore, in order to get an impression of the *possible* impact of the CDM abundance on the collider phenomenology in our model, we will analyze the case that the LSP does contribute to the CDM density, and apply a more loose bound of

$$\Omega_{\rm CDM} h^2 < 0.3 \; . \tag{26}$$

Notice that lower values than the ones permitted by (25) are naturally allowed if another particle than the lightest neutralino constitutes CDM. For our evaluation we have used the code MicroMegas [95, 96].
The prediction for M_h of **FUTB** with $\mu < 0$ is shown in fig. **??**. The constraints from the two B physics observables are taken into account. In addition the CDM constraint (evaluated with `Micromegas` [95, 96]) is fulfilled for the lighter (green) points in the plot, see ref. [81] for details. The lightest Higgs mass ranges in

$$M_h \sim 121 - 126 \text{ GeV}, \tag{27}$$

where the uncertainty comes from variations of the soft scalar masses, and from finite (i.e. not logarithmically divergent) corrections in changing renormalization scheme. To this value one has to add ± 3 GeV coming from unknown higher order corrections [89]. We have also included a small variation, due to threshold corrections at the GUT scale, of up to 5% of the FUT boundary conditions. Thus, taking into account the B physics constraints (and possibly the CDM constraints) results naturally in a light Higgs boson that fulfills the LEP bounds [85, 86].

In the same way the whole SUSY particle spectrum can be derived. The resulting SUSY masses for **FUTB** with $\mu < 0$ are rather large. The lightest SUSY particle starts around 500 GeV, with the rest of the spectrum being very heavy. The observation of SUSY particles at the LHC or the ILC will only be possible in very favorable parts of the parameter space. For most parameter combination only a SM-like light Higgs boson in the range of eq. (27) can be observed.

We note that with such a heavy SUSY spectrum the anomalous magnetic moment of the muon, $(g-2)_\mu$ (with $a_\mu \equiv (g-2)_\mu/2$), gives only a negligible correction to the SM prediction. However, since the SM is not regarded as excluded by $(g-2)_\mu$, we still see **FUTB** with $\mu < 0$ as the only surviving model. A more detailed analysis can be found in [81].

6. Concluding remarks on the Realistic Finite Unified Theories

The finiteness conditions in the supersymmetric part of the unbroken theory lead to relations among the dimensionless couplings, i.e. *gauge-Yukawa unification*. In addition the finiteness conditions in the SUSY-breaking sector of the theories lead to a tremendous reduction of the number of the independent soft SUSY-breaking parameters leaving one model (**A**) with three and another (**B**) with two free parameters. Therefore the *finiteness-constrained MSSM* consists of the well known MSSM with boundary conditions at the Grand Unification scale for its various dimensionless and dimensionful parameters inherited from the all-loop finiteness unbroken theories. Obviously these lead to an extremely restricted and, consequently, very predictive parameter space of the MSSM.

7. Unified Theories from Fuzzy Higher Dimensions

Coset Space Dimensional Reduction (CSDR) [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33] is a unification scheme for obtaining realistic particle models from gauge theories on higher D-dimensional spaces M^D. It suggests that a unification of the gauge and Higgs sectors of the Standard Model can be achieved in higher than four dimensions. Moreover the addition of fermions in the higher-dimensional gauge theory leads naturally, after CSDR, to Yukawa couplings in four dimensions. We present a study of the CSDR in the non-commutative context which sets the rules for constructing new particle models that might be phenomenologically interesting. One could study CSDR with the whole parent space M^D being non-commutative or with just non-commutative Minkowski space or non-commutative internal space. We specialize here to this last situation and therefore eventually we obtain Lorentz covariant theories on commutative Minkowski space. We further specialize to fuzzy non-commutativity, i.e. to matrix type non-commutativity. Thus, following [34, 35, 36], we consider non-commutative spaces like those studied in refs. [5, 6, 7, 8, 9, 2] and implementing the CSDR principle on these spaces we obtain the rules for constructing new particle models.

Next we reverse the above approach [37] and examine how a four dimensional gauge theory dynamically develops higher dimensions. The very concept of dimension therefore gets an extra, richer dynamical perspective. We present a simple field-theoretical model which realizes the above ideas. It is defined as a renormalizable $SU(N)$ gauge theory on four dimensional Minkowski space M^4, containing 3 scalars in the adjoint of $SU(N)$ that transform as vectors under an additional global $SO(3)$ symmetry with the most general renormalizable potential. We then show that the model dynamically develops fuzzy extra dimensions, more precisely a fuzzy sphere S^2_N. The appropriate interpretation is therefore as gauge theory on $M^4 \times S^2_N$. The low-energy effective action is that of a four dimensional gauge theory on M^4, whose gauge group and field content is dynamically determined by compactification and dimensional reduction on the internal sphere S^2_N. An interesting and rich pattern of spontaneous symmetry breaking appears, breaking the original $SU(N)$ gauge symmetry down to either $SU(n)$ or $SU(n_1) \times SU(n_2) \times U(1)$. The latter case is the generic one, and implies also a monopole flux induced on the fuzzy sphere. The values of n_1 and n_2 are determined dynamically.

We find moreover explicitly the tower of massive Kaluza-Klein modes corresponding to the effective geometry, which justifies the interpretation as a compactified higher-dimensional gauge theory. Nevertheless, the model is renormalizable.

A similar but different mechanism of dynamically generating extra dimensions has been proposed some years ago in [97], known under the name of "deconstruction". In this context, renormalizable four dimensional asymptotically free gauge theories were considered, which develop a "lattice-like" fifth dimension. This idea attracted considerable interest. Our model is quite different, and very simple: The $SU(N)$ gauge theory is shown to develop fuzzy extra dimensions through a standard symmetry breaking mechanism.

8. The Fuzzy Sphere

8.1. Ordinary and Fuzzy spherical harmonics

Let us start by recalling how to describe fields on the 2-sphere. The 2-sphere is a two-dimensional manifold embedded in $\mathbf{R^3}$, with a global $SO(3) \sim SU(2)$ isometry group, defined by the equation

$$x_1^2 + x_2^2 + x_3^2 = R^2 \tag{28}$$

for a coordinate basis $x_{\hat{a}}$ in $\mathbf{R^3}$. We define the coordinates $x_{\hat{a}}$ in terms of the spherical coordinates $y_a = (\theta, \phi)$ and radius R by,

$$x_1 = R \sin\theta \cos\phi, \tag{29}$$

$$x_2 = R \sin\theta \sin\phi, \tag{30}$$

$$x_3 = R \cos\theta, \tag{31}$$

which dictates the metric of the 2-sphere,

$$ds^2 = R^2\, d\theta^2 + R^2 \sin^2\theta\, d\phi^2. \tag{32}$$

The generators of $SU(2) \sim SO(3)$ are the angular momentum operators L_i,

$$L_{\hat{a}} = -i\varepsilon_{\hat{a}\hat{b}\hat{c}} x_{\hat{b}} \partial_{\hat{c}} \tag{33}$$

In terms of spherical coordinates the angular momentum operators are

$$L_1 = \quad i \sin\phi \frac{\partial}{\partial\theta} + i \cos\phi \cot\theta \frac{\partial}{\partial\phi}, \tag{34}$$

$$L_2 = -i \cos\phi \frac{\partial}{\partial\theta} + i \sin\phi \cot\theta \frac{\partial}{\partial\phi}, \tag{35}$$

$$L_3 = -i \frac{\partial}{\partial\phi}, \tag{36}$$

which we can summarize as

$$L_{\hat{a}} = -ik^a_{\hat{a}}\partial_a \tag{37}$$

The metric tensor can also be expressed in terms of the Killing vectors $k^a_{\hat{a}}$ (defined by the above equations) as

$$g^{ab} = \frac{1}{R^2}\, k^a_{\hat{a}} k^b_{\hat{a}}. \tag{38}$$

We can expand any function on the 2-sphere in terms of the eigenfunctions of the 2-sphere,

$$a(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta,\phi), \tag{39}$$

where a_{lm} is a complex coefficient and $Y_{lm}(\theta,\phi)$ are the spherical harmonics, which satisfy the equation

$$L^2 Y_{lm} = -R^2 \Delta_{S^2} Y_{lm} = l(l+1) Y_{lm}, \tag{40}$$

where Δ_{S^2} is the scalar Laplacian on the 2-sphere

$$\Delta_{S^2} = \frac{1}{\sqrt{g}}\,\partial_a(g^{ab}\sqrt{g}\,\partial_b). \tag{41}$$

The spherical harmonics have an eigenvalue $\mu \sim l(l+1)$ for integer $l = 0, 1, \ldots,$ with degeneracy $2l+1$. The orthogonality condition of the spherical harmonics is

$$\int d\Omega\, Y^{\dagger}_{lm} Y_{l'm'} = \delta_{ll'}\,\delta_{mm'}, \tag{42}$$

where $d\Omega = \sin\theta\, d\theta d\phi$.

The spherical harmonics can be expressed in terms of the cartesian coordinates $x_{\hat{a}}$ (with $\hat{a} = 1, 2, 3$) of a unit vector in $\mathbf{R^3}$,

$$Y_{lm}(\theta,\phi) = \sum_{\vec{a}} f^{(lm)}_{\hat{a}_1\ldots\hat{a}_l} x^{\hat{a}_1}\ldots x^{\hat{a}_l} \tag{43}$$

where $f^{(lm)}_{\hat{a}_1\ldots\hat{a}_l}$ is a traceless symmetric tensor of $SO(3)$ with rank l.

Similarly we can expand $N\times N$ matrices on a sphere as,

$$\hat{a} = \sum_{l=0}^{N-1}\sum_{m=-l}^{l} a_{lm}\hat{Y}_{lm} \tag{44}$$

$$\hat{Y}_{lm} = R^{-l}\sum_{\vec{a}} f^{(lm)}_{\hat{a}_1\ldots\hat{a}_l}\hat{x}^{\hat{a}_1}\ldots\hat{x}^{\hat{a}_l}, \tag{45}$$

where $\hat{x}_{\hat{a}} = \frac{2R}{\sqrt{N^2-1}} X_{\hat{a}}^{(N)}$ are the generators of $SU(2)$ in the N-dimensional representation and $f_{\hat{a}_1 \dots \hat{a}_l}^{(lm)}$ is the same tensor as in (43). The matrices $\hat{Y}_{lm}$ are known as fuzzy spherical harmonics for reasons explained in the next subsection. They obey the orthonormality condition

$$\mathrm{Tr}_N \left(\hat{Y}_{lm}^{\dagger} \hat{Y}_{l'm'} \right) = \delta_{ll'} \, \delta_{mm'}. \tag{46}$$

There is an obvious relation between equations (39) and (44), namely

$$\hat{a} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} a_{lm} \hat{Y}_{lm} \rightarrow \ a(\theta, \phi) = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi). \tag{47}$$

Notice that the expansion in spherical harmonics is truncated at $N-1$ reflecting the finite number of degrees of freedom in the matrix $\hat{a}$. This allows the consistent definition of a matrix approximation of the sphere known as fuzzy sphere.

8.2. The Matrix Geometry of the fuzzy sphere

According to the above discussion the fuzzy sphere [98, 2] is a matrix approximation of the usual sphere S^2. The algebra of functions on S^2 (for example spanned by the spherical harmonics) as explained in the previous section is truncated at a given frequency and thus becomes finite dimensional. The truncation has to be consistent with the associativity of the algebra and this can be nicely achieved relaxing the commutativity property of the algebra. The fuzzy sphere is the "space" described by this non-commutative algebra. The algebra itself is that of $N \times N$ matrices. More precisely, the algebra of functions on the ordinary sphere can be generated by the coordinates of $\mathbf{R}^3$ modulo the relation $\sum_{\hat{a}=1}^{3} x_{\hat{a}} x_{\hat{a}} = r^2$. The fuzzy sphere S_N^2 at fuzziness level $N-1$ is the non-commutative manifold whose coordinate functions $iX_{\hat{a}}$ are $N \times N$ hermitian matrices proportional to the generators of the N-dimensional representation of $SU(2)$. They satisfy the condition $\sum_{\hat{a}=1}^{3} X_{\hat{a}} X_{\hat{a}} = \alpha r^2$ and the commutation relations

$$[X_{\hat{a}}, X_{\hat{b}}] = C_{\hat{a}\hat{b}\hat{c}} X_{\hat{c}} \ , \tag{48}$$

where $C_{\hat{a}\hat{b}\hat{c}} = \varepsilon_{\hat{a}\hat{b}\hat{c}}/r$ while the proportionality factor α goes as N^2 for N large. Indeed it can be proven that for $N \to \infty$ one obtains the usual commutative sphere.

On the fuzzy sphere there is a natural $SU(2)$ covariant differential calculus. This calculus is three-dimensional and the derivations $e_{\hat{a}}$ along $X_{\hat{a}}$ of a function f are given by $e_{\hat{a}}(f) = [X_{\hat{a}}, f]$. Accordingly the action of the Lie derivatives on functions is given by

$$\mathcal{L}_{\hat{a}} f = [X_{\hat{a}}, f] \ ; \tag{49}$$

these Lie derivatives satisfy the Leibniz rule and the $SU(2)$ Lie algebra relation

$$[\mathcal{L}_{\hat{a}}, \mathcal{L}_{\hat{b}}] = C_{\hat{a}\hat{b}\hat{c}}\mathcal{L}_{\hat{c}}. \tag{50}$$

In the $N \to \infty$ limit the derivations $e_{\hat{a}}$ become $e_{\hat{a}} = C_{\hat{a}\hat{b}\hat{c}}x^{\hat{b}}\partial^{\hat{c}}$ and only in this commutative limit the tangent space becomes two-dimensional. The exterior derivative is given by

$$df = [X_{\hat{a}}, f]\theta^{\hat{a}} \tag{51}$$

with $\theta^{\hat{a}}$ the one-forms dual to the vector fields $e_{\hat{a}}$, $< e_{\hat{a}}, \theta^{\hat{b}} >= \delta^{\hat{b}}_{\hat{a}}$. The space of one-forms is generated by the $\theta^{\hat{a}}$'s in the sense that for any one-form $\omega = \sum_i f_i dh_i t_i$ we can always write $\omega = \sum_{\hat{a}=1}^{3} \omega_{\hat{a}}\theta^{\hat{a}}$ with given functions $\omega_{\hat{a}}$ depending on the functions f_i, h_i and t_i. The action of the Lie derivatives $\mathcal{L}_{\hat{a}}$ on the one-forms $\theta^{\hat{b}}$ explicitly reads

$$\mathcal{L}_{\hat{a}}(\theta^{\hat{b}}) = C_{\hat{a}\hat{b}\hat{c}}\theta^{\hat{c}} \ . \tag{52}$$

On a general one-form $\omega = \omega_{\hat{a}}\theta^{\hat{a}}$ we have $\mathcal{L}_{\hat{b}}\omega = \mathcal{L}_{\hat{b}}(\omega_{\hat{a}}\theta^{\hat{a}}) = \left[X_{\hat{b}}, \omega_{\hat{a}}\right]\theta^{\hat{a}} - \omega_{\hat{a}}C^{\hat{a}}{}_{\hat{b}\hat{c}}\theta^{\hat{c}}$ and therefore

$$(\mathcal{L}_{\hat{b}}\omega)_{\hat{a}} = \left[X_{\hat{b}}, \omega_{\hat{a}}\right] - \omega_{\hat{c}}C^{\hat{c}}{}_{\hat{b}\hat{a}} \ ; \tag{53}$$

this formula will be fundamental for formulating the CSDR principle on fuzzy cosets.

The differential geometry on the product space Minkowski times fuzzy sphere, $M^4 \times S^2_N$, is easily obtained from that on M^4 and on S^2_N. For example a one-form A defined on $M^4 \times S^2_N$ is written as

$$A = A_\mu dx^\mu + A_{\hat{a}}\theta^{\hat{a}} \tag{54}$$

with $A_\mu = A_\mu(x^\mu, X_{\hat{a}})$ and $A_{\hat{a}} = A_{\hat{a}}(x^\mu, X_{\hat{a}})$.

One can also introduce spinors on the fuzzy sphere and study the Lie derivative on these spinors. Although here we have sketched the differential geometry on the fuzzy sphere, one can study other (higher-dimensional) fuzzy spaces (e.g. fuzzy CP^M) and with similar techniques their differential geometry.

9. Dimensional Reduction of Fuzzy Extra Dimensions

9.1. Actions in higher dimensions seen as four-dimensional actions (Expansion in Kaluza-Klein modes)

First we consider on $M^4 \times (S/R)_F$ a non-commutative gauge theory with gauge group $G = U(P)$ and examine its four-dimensional interpretation.

$(S/R)_F$ is a fuzzy coset, for example the fuzzy sphere S^2_N. The action is

$$\mathcal{A}_{YM} = \frac{1}{4g^2}\int d^4x\, kTr\, tr_G\, F_{MN}F^{MN}, \tag{55}$$

where kTr denotes integration over the fuzzy coset $(S/R)_F$ described by $N \times N$ matrices; here the parameter k is related to the size of the fuzzy coset space. For example for the fuzzy sphere we have $r^2 = \sqrt{N^2-1}\pi k$ [2]. In the $N \to \infty$ limit kTr becomes the usual integral on the coset space. For finite N, Tr is a good integral because it has the cyclic property $Tr(f_1 \dots f_{p-1}f_p) = Tr(f_p f_1 \dots f_{p-1})$. It is also invariant under the action of the group S, that is infinitesimally given by the Lie derivative. In the action (55) tr_G is the gauge group G trace. The higher-dimensional field strength F_{MN}, decomposed in four-dimensional space-time and extra-dimensional components, reads as follows $(F_{\mu\nu}, F_{\mu\hat{b}}, F_{\hat{a}\hat{b}})$; explicitly the various components of the field strength are given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{56}$$

$$\begin{aligned} F_{\mu\hat{a}} &= \partial_\mu A_{\hat{a}} - [X_{\hat{a}}, A_\mu] + [A_\mu, A_{\hat{a}}], \\ F_{\hat{a}\hat{b}} &= [X_{\hat{a}}, A_{\hat{b}}] - [X_{\hat{b}}, A_{\hat{a}}] + [A_{\hat{a}}, A_{\hat{b}}] - C^{\hat{c}}{}_{\hat{a}\hat{b}}A_{\hat{c}}. \end{aligned} \tag{57}$$

Under an infinitesimal G gauge transformation $\lambda = \lambda(x^\mu, X^{\hat{a}})$ we have

$$\delta A_{\hat{a}} = -[X_{\hat{a}}, \lambda] + [\lambda, A_{\hat{a}}]\ , \tag{58}$$

thus F_{MN} is covariant under local G gauge transformations: $F_{MN} \to F_{MN} + [\lambda, F_{MN}]$. This is an infinitesimal abelian $U(1)$ gauge transformation if λ is just an antihermitian function of the coordinates $x^\mu, X^{\hat{a}}$ while it is an infinitesimal non-abelian $U(P)$ gauge transformation if λ is valued in Lie$(U(P))$, the Lie algebra of hermitian $P \times P$ matrices. In the following we will always assume Lie$(U(P))$ elements to commute with the coordinates $X^{\hat{a}}$. In fuzzy/non-commutative gauge theory and in Fuzzy-CSDR a fundamental role is played by the covariant coordinate,

$$\varphi_{\hat{a}} \equiv X_{\hat{a}} + A_{\hat{a}}\ . \tag{59}$$

This field transforms indeed covariantly under a gauge transformation, $\delta(\varphi_{\hat{a}}) = [\lambda, \varphi_{\hat{a}}]$. In terms of φ the field strength in the non-commutative directions reads,

$$F_{\mu\hat{a}} = \partial_\mu \varphi_{\hat{a}} + [A_\mu, \varphi_{\hat{a}}] = D_\mu \varphi_{\hat{a}}, \tag{60}$$

$$F_{\hat{a}\hat{b}} = [\varphi_{\hat{a}}, \varphi_{\hat{b}}] - C^{\hat{c}}{}_{\hat{a}\hat{b}}\varphi_{\hat{c}}\ ; \tag{61}$$

and using these expressions the action reads

$$\mathcal{A}_{YM} = \int d^4x\, Tr\, tr_G \left(\frac{k}{4g^2}F^2_{\mu\nu} + \frac{k}{2g^2}(D_\mu\varphi_{\hat{a}})^2 - V(\varphi) \right), \tag{62}$$

where the potential term $V(\varphi)$ is the $F_{\hat{a}\hat{b}}$ kinetic term (in our conventions $F_{\hat{a}\hat{b}}$ is antihermitian so that $V(\varphi)$ is hermitian and non-negative)

$$V(\varphi) = -\frac{k}{4g^2} Tr\, tr_G \sum_{\hat{a}\hat{b}} F_{\hat{a}\hat{b}} F_{\hat{a}\hat{b}}$$
$$= -\frac{k}{4g^2} Tr\, tr_G \left([\varphi_{\hat{a}}, \varphi_{\hat{b}}][\varphi^{\hat{a}}, \varphi^{\hat{b}}] - 4C_{\hat{a}\hat{b}\hat{c}} \varphi^{\hat{a}} \varphi^{\hat{b}} \varphi^{\hat{c}} + 2r^{-2} \varphi^2 \right). \quad (63)$$

The action (62) is naturally interpreted as an action in four dimensions. The infinitesimal G gauge transformation with gauge parameter $\lambda(x^\mu, X^{\hat{a}})$ can indeed be interpreted just as an M^4 gauge transformation. We write

$$\lambda(x^\mu, X^{\hat{a}}) = \lambda^\alpha(x^\mu, X^{\hat{a}}) \mathcal{T}^\alpha = \lambda^{h,\alpha}(x^\mu) T^h \mathcal{T}^\alpha \;, \quad (64)$$

where $\mathcal{T}^\alpha$ are hermitian generators of $U(P)$, $\lambda^\alpha(x^\mu, X^{\hat{a}})$ are $n \times n$ antihermitian matrices and thus are expressible as $\lambda(x^\mu)^{\alpha,h} T^h$, where T^h are antihermitian generators of $U(n)$. The fields $\lambda(x^\mu)^{\alpha,h}$, with $h = 1, \dots n^2$, are the Kaluza-Klein modes of $\lambda(x^\mu, X^{\hat{a}})^\alpha$. We now consider on equal footing the indices h and α and interpret the fields on the r.h.s. of (64) as one field valued in the tensor product Lie algebra Lie$(U(n)) \otimes$ Lie$(U(P))$. This Lie algebra is indeed Lie$(U(nP))$ (the $(nP)^2$ generators $T^h \mathcal{T}^\alpha$ being $nP \times nP$ antihermitian matrices that are linear independent). Similarly we rewrite the gauge field A_ν as

$$A_\nu(x^\mu, X^{\hat{a}}) = A_\nu^\alpha(x^\mu, X^{\hat{a}}) \mathcal{T}^\alpha = A_\nu^{h,\alpha}(x^\mu) T^h \mathcal{T}^\alpha, \quad (65)$$

and interpret it as a Lie$(U(nP))$ valued gauge field on M^4, and similarly for $\varphi_{\hat{a}}$. Finally $Tr\, tr_G$ is the trace over $U(nP)$ matrices in the fundamental representation.

Up to now we have just performed a ordinary fuzzy dimensional reduction. Indeed in the commutative case the expression (62) corresponds to rewriting the initial lagrangian on $M^4 \times S^2$ using spherical harmonics on S^2. Here the space of functions is finite dimensional and therefore the infinite tower of modes reduces to the finite sum given by Tr.

9.2. Non-trivial Dimensional reduction in the case of Fuzzy Extra Dimensions

Next we reduce the number of gauge fields and scalars in the action (62) by applying the Coset Space Dimensional Reduction (CSDR) scheme. Since $SU(2)$ acts on the fuzzy sphere $(SU(2)/U(1))_F$, and more in general the group S acts on the fuzzy coset $(S/R)_F$, we can state the CSDR principle in the same way as in the continuum case, i.e. the fields in the theory must be invariant under the infinitesimal $SU(2)$, respectively S, action up to an infinitesimal gauge transformation

$$\mathcal{L}_{\hat{b}} \phi = \delta_{W_{\hat{b}}} \phi = W_{\hat{b}} \phi, \quad (66)$$

$$\mathcal{L}_{\hat{b}} A = \delta_{W_{\hat{b}}} A = -DW_{\hat{b}}, \tag{67}$$

where A is the one-form gauge potential $A = A_\mu dx^\mu + A_{\hat{a}} \theta^{\hat{a}}$, and $W_{\hat{b}}$ depends only on the coset coordinates $X^{\hat{a}}$ and (like A_μ, A_a) is antihermitian. We thus write $W_{\hat{b}} = W_{\hat{b}}^{\alpha} \mathcal{T}^\alpha$, $\alpha = 1, 2 \ldots P^2$, where $\mathcal{T}^i$ are hermitian generators of $U(P)$ and $(W_b^i)^\dagger = -W_b^i$, here † is hermitian conjugation on the $X^{\hat{a}}$'s. In terms of the covariant coordinate $\varphi_{\hat{d}} = X_{\hat{d}} + A_{\hat{d}}$ and of

$$\omega_{\hat{a}} \equiv X_{\hat{a}} - W_{\hat{a}} \ , \tag{68}$$

the CSDR constraints assume a particularly simple form, namely

$$[\omega_{\hat{b}}, A_\mu] = 0, \tag{69}$$

$$C_{\hat{b}\hat{d}\hat{e}} \varphi^{\hat{e}} = [\omega_{\hat{b}}, \varphi_{\hat{d}}]. \tag{70}$$

In addition we have a consistency condition following from the relation $[\mathcal{L}_{\hat{a}}, \mathcal{L}_{\hat{b}}] = C_{\hat{a}\hat{b}}{}^{\hat{c}} \mathcal{L}_{\hat{c}}$:

$$[\omega_{\hat{a}}, \omega_{\hat{b}}] = C_{\hat{a}\hat{b}}{}^{\hat{c}} \omega_c, \tag{71}$$

where $\omega_{\hat{a}}$ transforms as $\omega_{\hat{a}} \to \omega'_{\hat{a}} = g \omega_{\hat{a}} g^{-1}$. One proceeds in a similar way for the spinor fields [34, 35, 36].

9.2.1. Solving the CSDR constraints for the fuzzy sphere

We consider $(S/R)_F = S_N^2$, i.e. the fuzzy sphere, and to be definite at fuzziness level $N-1$ ($N \times N$ matrices). We study here the basic example where the gauge group is $G = U(1)$. In this case the $\omega_{\hat{a}} = \omega_{\hat{a}}(X^{\hat{b}})$ appearing in the consistency condition (71) are $N \times N$ antihermitian matrices and therefore can be interpreted as elements of Lie$(U(N))$. On the other hand the $\omega_{\hat{a}}$ satisfy the commutation relations (71) of Lie$(SU(2))$. Therefore in order to satisfy the consistency condition (71) we have to embed Lie$(SU(2))$ in Lie$(U(N))$. Let T^h with $h = 1, \ldots, (N)^2$ be the generators of Lie$(U(N))$ in the fundamental representation, we can always use the convention $h = (\hat{a}, u)$ with $\hat{a} = 1, 2, 3$ and $u = 4, 5, \ldots, N^2$ where the $T^{\hat{a}}$ satisfy the $SU(2)$ Lie algebra,

$$[T^{\hat{a}}, T^{\hat{b}}] = C^{\hat{a}\hat{b}}{}_{\hat{c}} T^{\hat{c}} \ . \tag{72}$$

Then we define an embedding by identifying

$$\omega_{\hat{a}} = T_{\hat{a}}. \tag{73}$$

The constraint (69), $[\omega_{\hat{b}}, A_\mu] = 0$, then implies that the four-dimensional gauge group K is the centralizer of the image of $SU(2)$ in $U(N)$, i.e.

$$K = C_{U(N)}(SU((2))) = SU(N-2) \times U(1) \times U(1) \ ,$$

where the last $U(1)$ is the $U(1)$ of $U(N) \simeq SU(N) \times U(1)$. The functions $A_\mu(x,X)$ are arbitrary functions of x but the X dependence is such that $A_\mu(x,X)$ is Lie(K) valued instead of Lie$(U(N))$, i.e. eventually we have a four-dimensional gauge potential $A_\mu(x)$ with values in Lie(K). Concerning the constraint (70), it is satisfied by choosing

$$\varphi_{\hat{a}} = r\varphi(x)\omega_{\hat{a}} \ , \tag{74}$$

i.e. the unconstrained degrees of freedom correspond to the scalar field $\varphi(x)$ which is a singlet under the four-dimensional gauge group K.

The choice (73) defines one of the possible embedding of Lie$(SU(2))$ in Lie$(U(N))$. For example, we could also embed Lie$(SU(2))$ in Lie$(U(N))$ using the irreducible N-dimensional rep. of $SU(2)$, i.e. we could identify $\omega_{\hat{a}} = X_{\hat{a}}$. The constraint (69) in this case implies that the four-dimensional gauge group is $U(1)$ so that $A_\mu(x)$ is $U(1)$ valued. The constraint (70) leads again to the scalar singlet $\varphi(x)$.

In general, we start with a $U(1)$ gauge theory on $M^4 \times S^2_N$. We solve the CSDR constraint (71) by embedding $SU(2)$ in $U(N)$. There exist p_N embeddings, where p_N is the number of ways one can partition the integer N into a set of non-increasing positive integers [98]. Then the constraint (69) gives the surviving four-dimensional gauge group. The constraint (70) gives the surviving four-dimensional scalars and eq. (74) is always a solution but in general not the only one. By setting $\phi_{\hat{a}} = \omega_{\hat{a}}$ we obtain always a minimum of the potential. This minimum is given by the chosen embedding of $SU(2)$ in $U(N)$.

An important point that we would like to stress here is the question of the renormalizability of the gauge theory defined on $M_4 \times (S/R)_F$. First we notice that the theory exhibits certain features so similar to a higher-dimensional gauge theory defined on $M_4 \times S/R$ that naturally it could be considered as a higher-dimensional theory too. For instance the isometries of the spaces $M_4 \times S/R$ and $M_4 \times (S/R)_F$ are the same. It does not matter if the compact space is fuzzy or not. For example in the case of the fuzzy sphere, i.e. $M_4 \times S^2_N$, the isometries are $SO(3,1) \times SO(3)$ as in the case of the continuous space, $M_4 \times S^2$. Similarly the coupling of a gauge theory defined on $M_4 \times S/R$ and on $M_4 \times (S/R)_F$ are both dimensionful and have exactly the same dimensionality. On the other hand the first theory is clearly non-renormalizable, while the latter is renormalizable (in the sense that divergencies can be removed by a finite number of counterterms). So from this point of view one finds a partial justification of the old hopes for considering quantum field theories on non-commutative structures. If this observation can lead to finite theories too, it remains as an open question.

10. Dynamical Generation of Extra Dimensions

Let us now discuss a further development [37] of these ideas, which addresses in detail the questions of quantization and renormalization. This leads to a slightly modified model with an extra term in the potential, which dynamically selects a unique (nontrivial) vacuum out of the many possible CSDR solutions, and moreover generates a magnetic flux on the fuzzy sphere. It also allows to show that the full tower of Kaluza-Klein modes is generated on S^2_N.

10.1. The four-dimensional action

We start with a $SU(N)$ gauge theory on four dimensional Minkowski space M^4 with coordinates y^μ, $\mu = 0, 1, 2, 3$. The action under consideration is

$$\mathcal{S}_{YM} = \int d^4y\, Tr\left(\frac{1}{4g^2} F^\dagger_{\mu\nu} F_{\mu\nu} + (D_\mu\phi_a)^\dagger D_\mu\phi_a\right) - V(\phi) \tag{75}$$

where A_μ are $SU(N)$-valued gauge fields, $D_\mu = \partial_\mu + [A_\mu, .]$, and

$$\phi_a = -\phi_a^\dagger\,, \qquad a = 1, 2, 3$$

are 3 antihermitian scalars in the adjoint of $SU(N)$,

$$\phi_a \to U^\dagger \phi_a U$$

where $U = U(y) \in SU(N)$. Furthermore, the ϕ_a transform as vectors of an additional global $SO(3)$ symmetry. The potential $V(\phi)$ is taken to be the most general renormalizable action invariant under the above symmetries, which is

$$\begin{aligned} V(\phi) = Tr\ &(g_1\phi_a\phi_a\phi_b\phi_b + g_2\phi_a\phi_b\phi_a\phi_b - g_3\varepsilon_{abc}\phi_a\phi_b\phi_c + g_4\phi_a\phi_a) \\ &+ \frac{g_5}{N} Tr(\phi_a\phi_a)Tr(\phi_b\phi_b) + \frac{g_6}{N} Tr(\phi_a\phi_b)Tr(\phi_a\phi_b) + g_7. \end{aligned} \tag{76}$$

This may not look very transparent at first sight, however it can be written in a very intuitive way. First, we make the scalars dimensionless by rescaling

$$\phi'_a = R\,\phi_a,$$

where R has dimension of length; we will usually suppress R since it can immediately be reinserted, and drop the prime from now on. Now observe that for a suitable choice of R,

$$R = \frac{2g_2}{g_3},$$

the potential can be rewritten as

$$V(\phi) = Tr\left(a^2(\phi_a\phi_a + \tilde{b}\,\mathbb{1})^2 + c + \frac{1}{\tilde{g}^2} F^\dagger_{ab} F_{ab}\right) + \frac{h}{N}\, g_{ab} g_{ab} \tag{77}$$

for suitable constants $a, b, c, \tilde{g}, h$, where

$$\begin{aligned} F_{ab} &= [\phi_a, \phi_b] - \varepsilon_{abc}\phi_c = \varepsilon_{abc}F_c, \\ \tilde{b} &= b + \frac{d}{N} Tr(\phi_a \phi_a), \\ g_{ab} &= Tr(\phi_a \phi_b). \end{aligned} \tag{78}$$

We will omit c from now. Notice that two couplings were reabsorbed in the definitions of R and $\tilde{b}$. The potential is clearly positive definite provided

$$a^2 = g_1 + g_2 > 0, \qquad \frac{2}{\tilde{g}^2} = -g_2 > 0, \qquad h \geq 0,$$

which we assume from now on. Here $\tilde{b} = \tilde{b}(y)$ is a scalar, $g_{ab} = g_{ab}(y)$ is a symmetric tensor under the global $SO(3)$, and $F_{ab} = F_{ab}(y)$ is a $su(N)$-valued antisymmetric tensor field which will be interpreted as field strength in some dynamically generated extra dimensions below. In this form, $V(\phi)$ looks like the action of Yang-Mills gauge theory on a fuzzy sphere in the matrix formulation [99, 100, 101, 102]. It differs from the potential in (61) only by the presence of the first term $a^2(\phi_a\phi_a + \tilde{b})^2$, which is strongly suggested by renormalization. In fact it is necessary for the interpretation as pure YM action, and we will see that it is very welcome on physical grounds since it dynamically determines and stabilizes a vacuum, which can be interpreted as extra-dimensional fuzzy sphere. In particular, it removes unwanted flat directions.

10.2. Emergence of extra dimensions and the fuzzy sphere

The vacuum of the above model is given by the minimum of the potential (76). Finding the minimum of the potential is a rather nontrivial task, and the answer depends crucially on the parameters in the potential [37]. The conditions for the global minimum imply that ϕ_a is a representation of $SU(2)$, with Casimir $\tilde{b}$ (where it was assumed for simplicity $h = 0$). Then, it is easy to write down a large class of solutions to the minimum of the potential, by noting that any decomposition of $N = n_1N_1 + ... + n_hN_h$ into irreps of $SU(2)$ with multiplicities n_i leads to a block-diagonal solution

$$\phi_a = diag\Big(\alpha_1\, X_a^{(N_1)}, ..., \alpha_k\, X_a^{(N_k)}\Big) \tag{79}$$

of the vacuum equations, where α_i are suitable constants which will be determined below.

It turns out [37] that there are essentially only 2 types of vacua:

1. *Type I vacuum.* It is plausible that the solution (79) with minimal potential contains only representations whose Casimirs are close to $\tilde{b}$. In particular, let M be the dimension of the irrep whose Casimir $C_2(M) \approx \tilde{b}$ is closest to $\tilde{b}$. If furthermore the dimensions match as

$N = Mn$, we expect that the vacuum is given by n copies of the irrep (M), which can be written as $\phi_a = \alpha\, X_a^{(N)} \otimes 1_n$ with low-energy gauge group $SU(n)$.

2. *Type II vacuum.* Consider again a solution (79) with n_i blocks of size $N_i = \tilde{N} + m_i$, where $\tilde{N}$ is defined by $\tilde{b} = \frac{1}{4}(\tilde{N}^2 - 1)$, and assume that $\tilde{N}$ is large and $\frac{m_i}{\tilde{N}} \ll 1$. The action is then given by

$$V(\phi) = Tr\Big(\frac{1}{2\tilde{g}^2} \sum_i n_i\, m_i^2\, 1\!\!1_{N_i} + O(\frac{1}{N_i})\Big) \approx \frac{1}{2\tilde{g}^2}\, \frac{N}{k} \sum_i n_i\, m_i^2 \quad (80)$$

where $k = \sum n_i$ is the total number of irreps, and the solution can be interpreted in terms of "instantons" (nonabelian monopoles) on the internal fuzzy sphere [99]. Hence in order to determine the solution of type (79) with minimal action, we simply have to minimize $\sum_i n_i\, m_i^2$, where the $m_i \in \mathbb{Z} - \tilde{N}$ satisfy the constraint $\sum n_i\, m_i = N - k\tilde{N}$. In this case the the the solution with minimal potential among all possible partitions (79) is given by

$$\phi_a = \begin{pmatrix} \alpha_1\, X_a^{(N_1)} \otimes 1_{n_1} & 0 \\ 0 & \alpha_2\, X_a^{(N_2)} \otimes 1_{n_2} \end{pmatrix},$$

with low-energy gauge group $SU(n_1) \times SU(n_2) \times U(1)$.

Again, the $X_a^{(N)}$ are interpreted as coordinate functions of a fuzzy sphere S^2_N, and the "scalar" action

$$S_\phi = TrV(\phi) = Tr\Big(a^2(\phi_a\phi_a + \tilde{b})^2 + \frac{1}{\tilde{g}^2}\, F^\dagger_{ab} F_{ab}\Big)$$

for $N \times N$ matrices ϕ_a is precisely the action for a $U(n)$ Yang-Mills theory on S^2_N with coupling $\tilde{g}$, as shown in [99]. In fact, the new term $(\phi_a\phi_a + \tilde{b})^2$ is essential for this interpretation, since it stabilizes the vacuum $\phi_a = X_a^{(N)}$ and gives a large mass to the extra "radial" scalar field which otherwise arises. The fluctuations of $\phi_a = X_a^{(N)} + A_a$ then provide the components A_a of a higher-dimensional gauge field $A_M = (A_\mu, A_a)$, and the action can be interpreted as YM theory on the 6-dimensional space $M^4 \times S^2_N$, with gauge group depending on the particular vacuum. We therefore interpret the vacuum as describing dynamically generated extra dimensions in the form of a fuzzy sphere S^2_N. This geometrical interpretation can be fully justified by working out the spectrum of Kaluza-Klein modes. The effective low-energy theory is then given by the zero modes on S^2_N. This approach provides a clear dynamical selection of the geometry due to the term $(\phi_a\phi_a + \tilde{b})^2$ in the action.

Perhaps the most remarkable aspect of this model is that the geometric interpretation and the corresponding low-energy degrees of freedom depend in a nontrivial way on the parameters of the model, which are running under the RG group. Therefore the massless degrees of freedom and their geometrical interpretation depend on the energy scale. In particular, the low-energy gauge group generically turns out to be $SU(n_1) \times SU(n_2) \times U(1)$ or $SU(n)$, while gauge groups which are products of more than two simple components (apart from $U(1)$) do not seem to occur. The values of n_1 and n_2 are determined dynamically, and with the appropriate choice of parameters it is possible to construct vacuum solutions where they are as small, such as 2 and 3 [37].

It is interesting to examine the running of the coupling constants under the RG. R turns out to run only logarithmically, implies that the scale of the internal spheres is only mildly affected by the RG flow. However, $\tilde{b}$ is running essentially quadratically, hence is generically large. This is quite welcome here: starting with some large N, $\tilde{b} \approx C_2(\tilde{N})$ must indeed be large in order to lead to the geometric interpretation discussed above. Hence the problems of naturalness or fine-tuning appear to be rather mild here.

A somewhat similar model has been studied in [103, 104], which realizes deconstruction and a "twisted" compactification of an extra fuzzy sphere based on a supersymmetric gauge theory. Our model is different and does not require supersymmetry, leading to a much richer pattern of symmetry breaking and effective geometry. For other relevant work see e.g. [9].

The dynamical formation of fuzzy spaces found here is also related to recent work studying the emergence of stable submanifolds in modified IIB matrix models. In particular, previous studies based on actions for fuzzy gauge theory different from ours generically only gave results corresponding to $U(1)$ or $U(\infty)$ gauge groups, see e.g. [105, 106, 107] and references therein. The dynamical generation of a nontrivial index on noncommutative spaces has also been observed in [108, 109] for different models.

Our mechanism may also be very interesting in the context of the recent observation [110] that extra dimensions are very desirable for the application of noncommutative field theory to particle physics. Other related recent work discussing the implications of the higher-dimensional point of view on symmetry breaking and Higgs masses can be found in [111, 112, 113, 114]. These issues could now be discussed within a renormalizable framework.

11. Concluding remarks on the use of Fuzzy extra dimensions

Non-commutative Geometry has been regarded as a promising framework for obtaining finite quantum field theories and for regularizing quantum field theories. In general quantization of field theories on non-commutative spaces has turned out to be much more difficult and with less attractive ultraviolet features than expected see however ref. [115], and ref. [99, 116]. Recall also that non-commutativity is not the only suggested tool for constructing finite field theories. Indeed four-dimensional finite gauge theories have been constructed in ordinary space-time and not only those which are $\mathcal{N}=4$ and $\mathcal{N}=2$ supersymmetric, and most probably phenomenologically uninteresting, but also chiral $\mathcal{N}=1$ gauge theories [49, 50] which already have been successful in predicting the top quark mass and have rich phenomenology that could be tested in future colliders [49, 50, 38, 51, 74, 75, 76, 76, 117, 79]. In the present work we have not addressed the finiteness of non-commutative quantum field theories, rather we have used non-commutativity to produce, via Fuzzy-CSDR, new particle models from particle models on $M^4 \times (S/R)_F$.

A major difference between fuzzy and ordinary SCDR is that in the fuzzy case one always embeds S in the gauge group G instead of embedding just R in G. This is due to the fact that the differential calculus on the fuzzy coset space is based on $dimS$ derivations instead of the restricted $dimS - dimR$ used in the ordinary one. As a result the four-dimensional gauge group $H = C_G(R)$ appearing in the ordinary CSDR after the geometrical breaking and before the spontaneous symmetry breaking due to the four-dimensional Higgs fields does not appear in the Fuzzy-CSDR. In Fuzzy-CSDR the spontaneous symmetry breaking mechanism takes already place by solving the Fuzzy-CSDR constraints. The four-dimensional potential has the typical “mexican hat” shape, but it appears already spontaneously broken. Therefore in four dimensions appears only the physical Higgs field that survives after a spontaneous symmetry breaking. Correspondingly in the Yukawa sector of the theory we have the results of the spontaneous symmetry breaking, i.e. massive fermions and Yukawa interactions among fermions and the physical Higgs field. Having massive fermions in the final theory is a generic feature of CSDR when S is embedded in G [21]. We see that if one would like to describe the spontaneous symmetry breaking of the SM in the present framework, then one would be naturally led to large extra dimensions.

A fundamental difference between the ordinary CSDR and its fuzzy version is the fact that a non-abelian gauge group G is not really required in high dimensions. Indeed the presence of a $U(1)$ in the higher-dimensional theory is enough to obtain non-abelian gauge theories in four dimensions.

In a further development, we have presented a renormalizable four dimensional $SU(N)$ gauge theory with a suitable multiplet of scalars, which dynamically develops fuzzy extra dimensions that form a fuzzy sphere. The model can then be interpreted as 6-dimensional gauge theory, with gauge group and geometry depending on the parameters in the original Lagrangian. We explicitly find the tower of massive Kaluza-Klein modes, consistent with an interpretation as compactified higher-dimensional gauge theory, and determine the effective compactified gauge theory. This model has a unique vacuum, with associated geometry and low-energy gauge group depending only on the parameters of the potential.

There are many remarkable aspects of this model. First, it provides an extremely simple and geometrical mechanism of dynamically generating extra dimensions, without relying on subtle dynamics such as fermion condensation and particular Moose- or Quiver-type arrays of gauge groups and couplings, such as in [97] and following work. Rather, our model is based on a basic lesson from noncommutative gauge theory, namely that noncommutative or fuzzy spaces can be obtained as solutions of matrix models. The mechanism is quite generic, and does not require fine-tuning or supersymmetry. This provides in particular a realization of the basic ideas of compactification and dimensional reduction within the framework of renormalizable quantum field theory. Moreover, we are essentially considering a large N gauge theory, which should allow to apply the analytical techniques developed in this context.

In particular, it turns out that the generic low-energy gauge group is given by $SU(n_1) \times SU(n_2) \times U(1)$ or $SU(n)$, while gauge groups which are products of more than two simple components (apart from $U(1)$) do not seem to occur in this model. The values of n_1 and n_2 are determined dynamically. Moreover, a magnetic flux is induced in the vacua with non-simple gauge group, which is very interesting in the context of fermions, since internal fluxes naturally lead to chiral massless fermions [118]. This will be studied in detail elsewhere.

There is also an intriguing analogy between our toy model and string theory, in the sense that as long as $a = 0$, there are a large number of possible vacua (given by all possible partitions) corresponding to compactifications, with no dynamical selection mechanism to choose one from the other. Remarkably this analog of the "string vacuum problem" is simply solved by adding a term to the action.

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