Hecke surfaces and Duality transformations in Lattice Spin Systems

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Preface

In this paper I discuss some topics interesting for me long time. These themes relate with the following subjects:

1. Hecke surfaces and K- regular graphs.

2. Duality transformations for generalized Potts models. Each of them relates with deep mathematical and physical theories and they have nothing in common at the first sight. However, it become more evident in the last year that a deep internal relations between all these problems exist. Especially interesting and mysterious is the role of Hecke groups in this context. I consider only few examples of these topics. The paper is mainly expository.Some of the results are based on the paper jointly written with my colleagues. I would like to mention Robert Brooks, whose untimely death left without a remarkable friend and coauthor. His ideas of spectral characteristics of Laplacians on "typical" Riemann surfaces are currently not enough appreciated and then will be undoubtedly recognized.

1. The Basic Construction

Let Γ be a finite k-regular graph.

Definition 1. *An orientation $\mathcal{O}$ on Γ is an assignment for each vertex $v \in \Gamma$, of a cyclic ordering of the edges emanating from v.*

A graph $(\Gamma, \mathcal{O})$ with orientation is often referred to in the literature as a *fatgraph*.

Generalizing the construction of [2] for the case $k = 3$, we will associate to the oriented graph, $(\Gamma, \mathcal{O})$ a pair of Riemann surfaces, and $S^O(\Gamma, \mathcal{O})$ and

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$S^C(\Gamma, \mathcal{O})$. $S^O(\Gamma, \mathcal{O})$ will be a finite-area Riemann surface, and $S^C(\Gamma, \mathcal{O})$ will denote its conformal compactification. As in [2], the idea is that the spectral geometry of the non-compact surface $S^O(\Gamma, \mathcal{O})$ is controlled (up to geometric constants) by the spectral geometry of the oriented graph $(\Gamma, \mathcal{O})$, which may then be studied combinatorially. The spectral geometry of the closed surface $S^C(\Gamma, \mathcal{O})$ will be close to the spectral geometry of the open surface $S^O(\Gamma, \mathcal{O})$, provided that $S^O(\Gamma, \mathcal{O})$ satisfies *a large cusps condition*, which will be explained below.

A central part of the construction is the following.

Definition 2. *For given k, the Hecke group $\mathbf{H}_k$ is the discrete subgroup of $PSL(2, \mathbb{R})$ generated by the matrices*

$$A_k = \begin{pmatrix} 1 & 2\cos(\pi/k) \\ 0 & 1 \end{pmatrix} \qquad B_k = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A fundamental domain F_k for H_k is given by the region shown in Figure 1, where ρ_0 is the intersection in the upper half plane of the circles of radius 1 centered at 0 and $2\cos(\pi/k)$. Nothing that i is the fixed point of B_k, we see that ρ_0 is the fixed point of

$$A_k B_k = \begin{pmatrix} 2\cos(\pi/k) & -1 \\ 1 & 0 \end{pmatrix},$$

and hence

$$p_0 = \cos(\pi/k) + i\sin(\pi/k).$$

The corresponding circles meet at p_0 with angle $2\pi/k$.

The fact that A_k and B_k generate a discrete group can be read off from the Poincaré Polygon Theorem, the fact that A_k preserves the horocycle $y = 1$, and the fact that $A_k B_k$ is a rotation through angle $2\pi/k$ about p_0 and sends i to $2\cos(\pi/k) + i$.

In the particular case $k = 3$, we have $2\cos(\pi/k) = 1$, and we have the well-known generators and fundamental domain for $PSL(2, \mathbb{Z})$.

2. Hecke Surfaces

For each k, let $\mathcal{H}$ denote the collection of surfaces

$$\mathcal{H}_k = \{S : S = S^C(\Gamma, \mathcal{O}) \text{ for some k-regular } (\Gamma, \mathcal{O})\}.$$

Note that $\mathcal{H}_k$ is precisely the set $\mathcal{B}$ of *Belyi surfaces*, for which several characterizations are known, see [3, 4].

We will show:

Theorem 3. *For each k, $\mathcal{H}_k = \mathcal{B}$.*

It follows, for instance, that for any Riemann surface S and for any ε, there is a k-regular $(\Gamma, \mathcal{O})$ such that $S^C(\Gamma, \mathcal{O})$ is ε-close to S (for any reasonable metric on the moduli space of surfaces).

The point here is that the description of S as $S = S^C(\Gamma_k, \mathcal{O}_k)$ for some k may be very complicated, while for another k', the graph $\Gamma_{k'}, \mathcal{O}_{k'}$ might be quite simple.

Proof. By definition, a surface S lies in $\mathcal{B}$ if there is a holomorphic mapping

$$S \to S^2$$

with precisely three branch values.

Now suppose that $S = S^C(\Gamma, \mathcal{O})$ for a k-regular Γ. Then there are finitely many points $\{p_i\}$ on S such that

$$S^O = S - \cup\{p_i\} = \mathbb{H}/H,$$

where H is a finite-index subgroup of the Hecke group $\mathbf{H_k}$. Thus, S^O covers $\mathbb{H}/\mathbf{H_k}$, which is S^2 minus three singular points. This covering map extends over the points $\{p_i\}$ to give a holomorphic map $S \to S^2$ branched over three points. This shows that

$$\mathcal{H}_k \subset \mathcal{B}.$$

To show the other direction, we begin by considering the k-regular oriented graph $(\Gamma_{2,k}, \mathcal{O}_{2,k})$ on two vertices v_1 and v_2. The cyclic ordering at v_1 is the cyclic ordering $0, 1, \ldots, k-1)$, while the cyclic ordering at v_2 is the inverse ordering $0, k-1, \ldots, 1)$. Each LHT path is of length 2.

It is then easy to calculate that $S^0(\Gamma_{2,k}, \mathcal{O}_{2,k})$ is S^2 with punctures at the k points $e^{2\pi i l/k}$.

Thus, S will lie in $\mathcal{H}_k$ if there are finitely many points $\{p_i\}$ such that $S \bigcup \{p_i\}$ covers $S^O(\Gamma_{2,k}, \mathcal{O}_{2,k})$. But if S is a Belyi surface, there is a holomorphic map $\phi : S \to S^2$ branched over three points, which we may take to be $1, e^{2\pi i/k}, e^{4\pi i/k}$. Deleting from S the inverse image under ϕ of these points and also the inverse images of the other points $e^{2\pi l/k}$ exhibits S minus a finite number of points as a cover of $S^O(\Gamma_{2,k}, \mathcal{O}_{2,k})$.

This shown that

$$\mathcal{B} \subset \mathcal{H}_k,$$

completing the proof of the theorem. □

We would like to mention that the result of the theorem is not new. Some similar realization of Belyi surfaces see in [8, 7]. We give only the new and simple proof, related to the graph theory.

3. Riemann surfaces $S^O(\Gamma)$ and$S^C(\Gamma)$

In this section we describe how to read off some geometric properties of the surfaces $S^O(\Gamma)$ and $S^C(\Gamma)$ from the combinatorics of the graph $\Gamma(G, \mathcal{O})$.

Definition 4. *A left-hand -turn path (LHT) on $\Gamma(G, \mathcal{O})$. is a closed path on Γ such that, at each vertex, the path turns left in the orientation O.*

Traveling on a path on Γ which always turn left describes a path on $S^C(\Gamma, \mathcal{O})$ which travels around a cusp . Let $l = l(\Gamma(G), \mathcal{O})$ to be the number of disjoint LHT paths , then the topology of $S^O(\Gamma, \mathcal{O})$ is describable in terms of l and the number of vertices $2n$. The graph Γ divides $S^O(\Gamma, \mathcal{O})$ into l regions , each bordered by a LHT path and containing one cusp in interior. $(\Gamma, \mathcal{O})$ Using the Euler characteristic formula : $\chi(S^O(\Gamma, \mathcal{O}) = 2\mathrm{n} - l\mathrm{n} + l = 2 - 2g$. So the genus g($S^O(\Gamma, \mathcal{O})$) is given by $g = 1 + (n - l)/2$ and the number of cusps is l.

Remark 5. *The topology of $S^O(\Gamma, \mathcal{O})$is heavily dependent on the choice of orientation Ω.*

Example 6. *[2] The usual orientation on the 3-regular graph which is the 1-skeleton of cube contains six LHT paths, giving the associate surface of sphere with six punctures, while a choice on this can have either two,four or six LHT paths, so that the the associated surface can have genus $0, 1, 2$.*

Example 7. Platonic solids. *Let π_k be the k-th Platonic graph of [4]. It is the k-regular graph defined by*

$$\{(a, b) \in \mathbb{Z}/k \times \mathbb{Z}/k, a, b \quad \textit{relatively prime to } k\}/(a, b) \sim (-a, -b).$$

Two vertices (a, b) and (c, d) are joined by an edge provided that

$$det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm 1 (mod\ k).$$

An orientation $\mathcal{O}$ on π_k may be defined as follows: at the vertex (a, b), let $< (a, b), (c, d) >$ be an edge. We choose the sign of $c, d)$ so that

$$det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv 1 (mod\ k).$$

Then the next edge in the cyclic order at (a, b) is $< (a, b), (c - a), (d - b) >$.

With this orientation all LHT paths are of length 3, by virtue of the sequence

$$< (a, b), (c, d) > \to < (c - a, d - b), (-a, -b) > \to < (c, d), (a - c, b - d) > .$$

sequence

$$< (a, b), (c, d) > \to < (c - a, d - b), (-a, -b) > \to < (c, d), (a - c, b - d) > .$$

The surface $S^C(\pi_k, \mathcal{O})$ is the *Platonic surface* P_k, which is the compactification of the modular surface

$$S_k = \mathbb{H}^2/\Gamma_k, \qquad \Gamma_k = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\ k) \right\}.$$

Example 8. *Let* $(\Gamma, \mathcal{O})$ *be an oriented* k*-regular graph, all of whose LHT paths have length* l*. Then we may define the dual oriented graph* $(\Gamma^*, \mathcal{O}^*)$ *as an* l*-regular graph, all of whose LHT paths are of length* k*, as follows: the vertices of* Γ^* *are the LHT paths of* Γ*. The edges* $\{e\}$ *correspond to the edges* e *of* Γ*, and* $\{e\}$ *joins the two (not necessarily distinct) LHT paths to which* e *belongs.*

The orientation $\mathcal{O}^*$ *on* Γ *is given as follows: given a LHT path* γ *and an edge* e *on* Γ*, the next element in the cyclic ordering at* $\{\gamma\}$ *is* $\{e'\}$*, where* e' *is the edge following* e *along the path* γ*.*

Theorem 9.

$$S^C(\Gamma, \mathcal{O}) = S^C(\Gamma^*, \mathcal{O}^*).$$

This duality concerns with two types of compactification of surface by horocycles and to add the points of absolute by geodesics going to cusps.

The point here is that it may be that $(\Gamma, \mathcal{O})$ is difficult to analyze, but $(\Gamma^*, \mathcal{O}^*)$ may be relatively easy to understand. For instance, bounding the Cheeger constant and first eigenvalue of the dual Platonic graphs π_k^* uniformly from below is equivalent to Selberg's Theorem [10] up to constant, but the Cheeger constant and first eigenvalue of π_k^* may be calculated in an elementary manner [**?**].

4. Large Cusps

The geometry of the cusps can also be read off from $\Gamma(G, \mathcal{O})$. In [6] R. Brooks suggested the following construction.

Definition 10. *Let* S^O *be a finite area Riemann surface.* S^O *has cusps of length* $\gg L$ *(shortly large cusps conditions) if there is a system* c_i *of closed horocycles such that :*

i) Each horocycle has length at least L*,*

ii) Each cusp is contained in the interior of one of the c_i *,*

iii)The interior of the c_i *are disjoint.*

The importance of this conditions follows from the theorem that asserts:

When S^O satisfies the large cusps conditions, the spectral geometry of S^O and S^C are close. See the exact statement in [6]

We give an outline of the proof. If each cusp has a horocycle of length at least 2π, than you can close off the cusp with a metric of negative curvature by changing the metric conformally inside the cusp. The number 2π arises as necessary condition for this by Gauss-Bonnet theorem. D. Mangoubi

[14] shows it is sufficient. The corresponding number for k-regular graphs would be the first integer m such that $2m\cos(\pi/k) > 2\pi$. So the limiting behavior as k is going to infinity is $m = 4$ and for $k = 3$ (modular group), $m = 7$. In particular, Mangoubi calculate how long the cusps must be to guarantee that S^C carries a metric of negative curvature. He shows that this will be the case provided that the cusps have length $\geq 2\pi$.

We remark that the large cusps condition *does not imply* that all the closed paths on the graph Γ are short. It is a condition only on the LHT paths. Thus, the oriented graph $(\Gamma, \mathcal{O})$ may have plenty of short geodesics, while still having cusps of length $\geq L$ for some large L.

Of course, it is not always convenient to change the metric within closed horocycles. For instance, the Platonic graphs π_k have LHT paths all of length 3, and so do not have large cusps. In [14] it is shown by example that one cannot weaken the large cusps condition by, for instance, replacing horocycles with a general condition such as large geodesic curvature and convexity. However, in special cases we may still modify the metric on $S^O(\Gamma, \mathcal{O})$ in a canonical way to obtain the desired results. Here is an example geared to handle the Platonic graphs:

Theorem 11. *There exists a k_0 and a number d_0 with the following property: let $(\Gamma, \mathcal{O})$ be a k-regular graph, for $k \geq k_0$, such that all the LHT paths have length equal to 3. Then there exist neighborhoods of the cusps and of the vertices of $S^O(\Gamma, \mathcal{O})$ and $S^C(\Gamma, \mathcal{O})$, depending only on k_0, such that outside of these neighborhoods, the metrics ds_C^2 and ds_O^2 satisfy*

$$\frac{1}{d_0} ds_O^2 \leq ds_C^2 \leq d_0 ds_O^2.$$

The notation is meant to emphasize that we do not have $d_0 \to 0$ as $k \to \infty$.

5. Geodesics on Graphs and surfaces

The geodesics of $S^O(\Gamma, \mathcal{O}))$ is possible to describe in terms of $(\Gamma, \mathcal{O})$. Let $\mathfrak{L} =$ and $\mathfrak{R} =$ A closed path P of length k on the graph may be described by starting at a midpoint of an edge, and then giving a sequence $(w_1, .., w_n)$, where each w_i is either l or r, signifying a left or right turn at the upcoming vertex. Let $M_p = W_1 \cdots W_k$, where $W_j = \mathfrak{L}$ if $w_j = l$ and $W_j = \mathfrak{R}$ if $w_j = r$. The closed path P on Γ is then homotopic to a closed geodesic $\gamma(P)$ on $S^O(\Gamma, \mathcal{O}))$ whose length $\gamma(P)$ is given by

The length $\gamma(P)$ depends strongly on ω. For instance, if the path P contains only left hand turns then length $\gamma(P)$ =0. If the path$P$ of length r consists of alternating left and right hand turns, then length$(\gamma(P))$ $=r\log(\frac{3+\sqrt{5}}{2})$.

6. The generalized Potts models

In this section I consider some relations of Hecke groups with Potts models. I briefly remind the definition of classical Potts model in the planar case. I refer for details to the book of R. Baxter [5]

A. Z_n Potts model Let L be a two-dimensional lattice. With each site i we associate a "spin" σ_i which takes n values. Two adjacent spins σ_i and σ_j interact with the energy $-J\delta(\sigma_i, \sigma_j))$ where $\delta(,)$ is the usual Dirac $\delta(,)$ -function. The total energy is

$$E = -J \sum_{(i,j)} \delta(\sigma_i, \sigma_j)) \tag{1}$$

where the the summation is over all edges(i, j) of L. The partition function is

$$Z_n = \sum_{\sigma} \exp\{K \sum_{(i,j)} \delta(\sigma_i, \sigma_j))\} \tag{2}$$

Here the summation is over all values of spin$\sigma(_i)$.

Remark 12. *The Potts model is possible to determine on any graph L.*

In 1969 P. W. Kasteleyn and C. M. Fortuin have found that $Z - n$ Potts model can be expressed as a dichromatic polynomial ,known in the graph theory (H.Whitney,T.Tutte). We set v=$exp(K) - 1$. Consider a typical graph G containing l bonds and c connected components(including isolated sites). Let E be the number of edges of the graph L. Then the summand in (1) is the sum of two terms 1 and $v\delta(i, j)$. So the product can be expanded as the sum of 2^E terms . Each of these 2^E terms can be associated with a bond-graph on L. Then the corresponding term in the expansion contains factor v^l. Summing over independent spins and over all components we obtain the contribution of these terms $n^c v^l$.So the partition function Z_n= (2). The summation is over all graphs G drawn on L.

The expression (2) is called a dichromatic polynomial or Whitney-Tutte polynomial.

In the anti ferromagnetic case $K = -\infty$ and $v = -1$. $Z_n = \sum q^C(-1)^l = P_n(q)$ reduces to chromatic polynomial.

It is clear that $P_n(q)$ determines the number of ways of coloring the sites of L with q colors ,no two adjacent sites having the same color. So $P_n(q)$ is the polynomial in q, which coincides with partition function Z_n (2) with $v = -1$.

Remark 13. *It is important to mention that the expression (2) is determined for any complex numbers q, not necessary integers. There is a beautiful conjecture concerning the behavior of zeros of chromatic polynomials.*

Conjecture 14. *Let us consider a chromatic polynomial $P_n(q)$ for arbitrary large planar graph. Then the real zeros of $P_n(q)$ cluster round limit points. These limit points are so called"Beraha numbers" $q = [2\cos(\pi/k)]^2$, $k = 2, 3..$*

This conjecture is still unproved. There is an interesting approach using quantum groups [10] . I would like to outline another approach using Hecke graphs.In this case it is necessary to consider the Cayley graph generating by Hecke groups. The partition function of Potts model determined on this graph reduces to the chromatic polynomials with desire properties.

Remark 15. *We mention at the end that the famous problem of four colors on a planar graph is exactly equivalent to the property that $P_n(4)$ is always equal zero.*

7. Generalized Potts model

In the paper[12, 13] we generalized Kramers-Wannier (KW) duality to the case where spins take values in any finite groups , not necessary abelian. In the paper[13] it was generalized to compact groups. The idea of these papers is to study instead of of the relation of the group G, where spins take it values , and the dual object $\hat{G}$ the pairs of two algebras , namely the group algebra$[G]$ and the space of regular functions on G,$C(G)$.This approach is very natural in the spirit of quantum groups. From this point of view is interesting to study the so called McKay correspondence which attached to any finite group K of $SU(2)$ a certain graph which coincides with affine extensions of Dynkin diagrams . Recently these results were extended by I.Dolgachev (in Press) to the cocompact discrete subgroups γ of $SU(1,1)$. It is interesting problem to consider McKay correspondence to Hecke groups. Remark 3. The last which I would like only to mention is the relation of Hecke groups with the two-dimensional quantum field theory. These groups appeared as the monodromy representations of some colored braid groups and determined the correlation functions in Z_3 and parafermionic Potts models.[1, 9, 15]

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