Cohomological products via intersections on Morse – Floer moduli spaces*

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Abstract

Multiplications in cohomology of a smooth manifold can be described in terms of Morse–Floer gradient lines on certain graphs or Riemann surfaces with boundary. This description is functorial with respect to the variation of parameters.

1. Introduction

It is well known that the intersection theory of smooth manifolds is cohomological in its nature. The most basic, classical, example is the formula

$$R \cdot S = \int_M PD(R) \cup PD(S), \tag{1}$$

where R, S are closed smooth submanifolds of a closed smooth manifold M, $R \cdot S$ their (signed) intersection number, $PD(R), PD(S)$ their Poincaré duads, and $\cup$ their cup (or wedge, in terms of differential forms) product (see for example [4] for details).

The main result in Morse–Floer theory is that the (co)homology of a smooth manifold is encoded in a moduli space of solutions of the system

$$\frac{d\gamma}{dt} = -\nabla f(\gamma) \tag{2}$$

in Morse, or

$$\bar{\partial}_J u = -\nabla H(u), \tag{3}$$

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where

$$\bar{\partial}_J u := \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t}$$

in Floer case. Here $f : M \to \mathbb{R}$ is a generic Morse function, $H : T^*M \to \mathbb{R}$ is a generic compactly supported Hamiltonian, $J : T^*M \to T^*M$ is a generic almost complex structure compatible with standard symplectic form on a cotangent bundle, and $\bar{\partial}_J$ is a Cauchy–Riemann operator defined in terms of J (see [14] for more precise definitions). If a domain of γ is a real line $\mathbb{R}$, and a domain of u is a strip $[0,1]\times\mathbb{R}$, the equations (2) (resp. (3)) define the (co)boundary operator on Morse (resp. Floer) (co)homology chain groups (see [11, 16] and references given there). If a domain of γ is a more complicated graph, and a domain of u a more general Riemann surface with boundary, the equations (2) and (3) define various cohomological products. In particular, if a domain of γ is a tree with 3 vertices at the boundary, and 3 edges parametrized by $(-\infty,0]$, $(-\infty,0]$, $[0,+\infty)$, then (2) describes the intersection of two unstable and one stable manifolds of ∇f. It is not difficult to see that it follows from (1) that (2) then defines a standard cup product

$$\cup : H^r(M)\otimes H^s(M) \to H^{r+s}(M).$$

For more general cohomology operations via graphs and Morse gradient flows we refer the reader to [3] (see also [13, 15] for generalizations and applications to Lagrangian submanifolds theory).

It turns out that these products are functorial with respect to variations of parameters (Morse functions and Hamiltonians).

2. Piunikhin-Salamon-Schwarz isomorphism

Let M be a smooth manifold, $f : M \to \mathbb{R}$ a Morse function on M, and $H : T^*M \to \mathbb{R}$ a compactly supported smooth Hamiltonian on a cotangent bundle. It is known that the Morse homology groups $HM_*(f)$ are isomorphic to the Floer homology groups $HF_*(H)$. More precisely, let $CM_p(f)$ be the vector space over $\mathbb{Z}_2$, generated by critical points of f with Morse index p, and let $CF_p(H)$ be the vector space over $\mathbb{Z}_2$, generated by critical points of the action functional

$$\mathcal{A}_H(\zeta) = \int_0^1 pdq - Hdt,$$

on the space of smooth paths

$$\zeta : [0,1] \to T^*M$$

with boundary conditions

$$\zeta(0), \zeta(1) \in 0_M,$$

where 0_M is the zero section of T^*M. Here p is the Maslov index for paths (see [20, 21]). Then the negative gradient equations (2) for

$$\gamma : \mathbb{R} \to M$$

define the Morse boundary operator

$$\partial_M : CM_p(f) \to CM_{p-1}(f).$$

More precisely, if $n(x_-, x_+)$ denotes the number of solutions of (2) with the asymptotic conditions $\gamma(\pm\infty) = x_\pm$ then ∂_M is given, on a chain level, by

$$\mathrm{Crit}_p(f) \ni x \mapsto \sum_y n(x,y)y \in C_{p-1}(f). \tag{4}$$

In a completely analogous way the nonlinear nonhomogeneous Cauchy–Riemann equations (3) for

$$u : \mathbb{R} \times [0,1] \to T^*M, \quad u(\{0,1\} \times \mathbb{R}) \subset 0_M$$

define the Floer boundary operator

$$\partial_F : CF_p(H) \to CF_{p-1}(H).$$

Note that these constructions make use of zero–dimensional components of solutions of (2) and (3), since only in this case $n(x,y)$ is a finite number. The higher dimensional components of solutions play role in homotopy theory [2] or in generalized (co)homology [17].

Morse cohomology groups $HM^*(f)$ are defined a dual way, by taking a cochain complex $CM^*(f)$ to be $\mathrm{Hom}(CM_*(f), \mathbb{Z}_2)$ and the coboundary operator

$$\delta : CM^p(f) \to CM^{p+1}(f)$$

defined by

$$\langle \delta a, \alpha \rangle := \langle a, \partial\alpha \rangle.$$

There is an obvious isomorphism

$$\Gamma : CM_*(f) \xrightarrow{\cong} CM_{n-*}(-f)$$

which is compatible with the boundary operators in homology and thus gives rise to the isomorphism

$$\sigma_f : HM_{n-*}(-f) \xrightarrow{\cong} HM^*(f). \tag{5}$$

Similar construction is easily carried out in Floer case.

The isomorphism

$$HM_*(f) \cong HF_*(H)$$

established by Floer [5] uses the fact that $HM_*(f)$ and $HF_*(H)$ do not depend on f and H, and the convenient choice of C^2–small autonomous Hamiltonian H.

In [10] we proved that there exists an isomorphism that is functorial with respect to the variation of f and H. More precisely, there exist isomorphisms

$$\Phi : HF_*(H) \to HM_*(f), \qquad \Psi : HM_*(f) \to HF_*(H)$$

such that the diagram

$$\begin{array}{ll} HF_*(H^\alpha) & \xrightarrow{S^{\alpha\beta}} HF_*(H^\beta) \\ \uparrow \Psi^\alpha & \qquad \uparrow (\Phi^\beta)^{-1} \\ HM_*(f^\alpha) & \xrightarrow{T^{\alpha\beta}} HM_*(f^\beta) \end{array}$$

commutes. These isomorphisms are constructed as the intersections of the solutions of (2) and (3) in appropriate moduli spaces or, more explicitly, by means of solutions of systems

$$\begin{cases} \frac{d\gamma}{dt} = -\nabla f(\gamma(t)), \\ \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = -\nabla(\rho_R H)(u) \\ u(s,0), u(s,1) \in O_M, \\ \gamma(-\infty) = p, \\ u(+\infty, t) = x(t), \\ \gamma(0) = u(0, \frac{1}{2}) \end{cases} \qquad \begin{cases} \frac{d\gamma}{dt} = -\nabla f(\gamma(t)), \\ \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = -\nabla(\chi_R H)(u) \\ u(s,0), u(s,1) \subset O_M, \\ \gamma(+\infty) = p, \\ u(-\infty, t) = x(t), \\ \gamma(0) = u(0, \frac{1}{2}) \end{cases}$$

where $\rho_R, \chi_R : \mathbb{R} \to \mathbb{R}$ are conveniently chosen smooth functions with compact supports. The isomorphisms Φ and Ψ are then constructed in a similar way as the boundary operators (4) (see [10], and also [8] for more details, [19] for the periodic orbits case, [9] for generalizations from $\mathbb{Z}_2$ to more general coefficients ring, and [1] for generalizations from T^*M to more general symplectic manifolds).

By duality, we get the same results in cohomology.

3. Products in Morse–Floer cohomology

If the domain for γ in (2) is a graph Γ, the equations (2) for $\gamma : \Gamma \to M$ give rise to various (for different Γs) operations in Morse (co)homology (see [3]).

Similarly, let Σ be a Riemann surface with boundary $\partial\Sigma$ and several asymptotic ends. More exactly, assume that Σ is an open subset of a complex plane with $m + k$ cylindrical ends Σ_j, i.e. subsets that are for fixed $R > 0$ the images of injective holomorphic mappings

$$\phi_i : (-\infty, R] \times [0,1] \to \Sigma,\ i = 1, \ldots, m;$$
$$\phi_j : [R, +\infty) \times [0,1] \to \Sigma,\ j = m+1, \ldots, m+k$$

and let $\Sigma_0 := \Sigma \setminus \cup_{j=1}^{m+k}\Sigma_j$ be homeomorphic to a disc. Then the equations (3) for $u : \Sigma \to T^*M$, $u(\partial\Sigma) \subset 0_M$ define (co)homological operation in Floer (co)homology.

For simplicity, let us describe the construction for $m = 2$ in Morse case in more details. In this case Γ is a tree with three edges. We identify two edges e_1, e_2 with $(-\infty, 0]$ and one e_3 with $[0, +\infty)$. Let f_1, f_2, f_3 be Morse functions on M and let $g_i := -f_i$. Consider the mapping $I : T \to M$ such that $\gamma_i := I|_{e_i}$, $i = 1, 2, 3$, are the solutions of

$$\begin{cases} \frac{d\gamma_i}{ds} = -\nabla(g_i)(\gamma_i) \\ \gamma_i(-\infty) = p_i, \ i = 1, 2 \\ \gamma_3(+\infty) = p_3. \end{cases} \tag{6}$$

Here p_i are critical points of g_i. If we denote by $\vec{g} = (g_1, g_2, g_3)$, $\vec{p} = (p_1, p_2, p_2)$ and by $\mathcal{M}(\vec{p}, \vec{g})$ solutions of (6) then

$$\mathcal{M}(\vec{p}, \vec{g}) = W^u(g_1, p_1) \cap W^u(g_2, p_2) \cap W^s(g_3, p_3)$$

where $W^u(g, p)$ (respectively $W^s(g, p)$) denotes the unstable (respectively stable) manifold of a critical point p of a Morse function g. For generic choices $\mathcal{M}(\vec{p}, \vec{g})$ is a smooth manifold, and for the appropriate choice of Morse indices it is zero dimensional. Denote by $n(\vec{p}, \vec{g})$ the number of solutions of (6) (i.e. the cardinality of $\mathcal{M}(\vec{p}, \vec{g})$) mod 2. Let us define Ψ_M by

$$\Psi_M(\vec{g}) = \sum_{\vec{p}} n(\vec{p}, \vec{g}) p_1 \otimes p_2 \otimes p_3 \in CM_{n-k}(g_1) \otimes CM_{n-l}(g_2) \otimes CM_{n-(k+l)}(g_3).$$

Then

$$\Psi_M(\vec{g}) \in HM_{n-k}(g_1) \otimes HM_{n-l}(g_2) \otimes HM_{n-(k+l)}(g_3).$$

Let $[a_1] \in HM^k(f_1) \cong HM^{n-k}(g_1)$ and $[a_2] \in HM^l(f_2) \cong HM^{n-l}(g_2)$ be cohomological class of $a_1 \in CM^k(f_1)$, $a_2 \in CM^l(f_2)$. Then there exist a contraction

$$\langle a_1 \otimes a_2, \Psi_M(\vec{g}) \rangle \in CM_{n-(k+l)}(g_3).$$

Now we define

$$[a_1] \cup_M [a_2] := [\sigma_{f_3}(\langle a_1 \otimes a_2, \Psi_M(\vec{g}) \rangle)] \in HM^{k+l}(f_3).$$

Using cobordism arguments, one can show that this definition does not depend on the choice of cycles a_1 and a_2 within the same cohomological classes. Similar construction gives rise to cup product $\cup_F$ in Floer cohomology (see [18] for the details).

Denote the general (for arbitrary m) operations defined in this way by $\mathcal{O}_M$ and $\mathcal{O}_F$. It turns out that then the diagram

$$\begin{array}{ccc} HF^{k_1}(H_1) \otimes \ldots \otimes HF^{k_m}(H_m) & \xrightarrow{\mathcal{O}_F} & HF^{k_1+\ldots+k_m}(H_{m+1}) \\ \Psi_1 \otimes \ldots \otimes \Psi_m \uparrow & & \Psi_{m+1} \uparrow \\ HM^{k_1}(f_1) \otimes \ldots \otimes HM^{k_m}(f_m) & \xrightarrow{\mathcal{O}_M} & HM^{k_1+\ldots+k_m}(f_{m+1}). \end{array}$$

commutes [12].

For $m = 2$ the above diagram becomes

$$\begin{array}{ccc} HF^k(H_1) \otimes HF^l(H_2) & \xrightarrow{\cup_F} & HF^{k+l}(H_3) \\ \tau_1 \otimes \tau_2 \uparrow & & \tau_3 \uparrow \\ HM^k(f_1) \otimes HM^l(f_2) & \xrightarrow{\cup_M} & HM^{k+l}(f_3). \end{array}$$

The commutativity of this diagram was established in [22].

The proofs of the above facts are based on the detailed analysis of spaces of intersections of solutions of (2) on graphs and (3) on Riemann surfaces, in certain moduli spaces. More precisely, the reasons behind the commutativity of the above diagrams is certain regularity of divergence of sequences of mappings

$$\gamma_i : (-\infty, 0] \to M,\ i = 1, \dots, m,$$
$$\gamma_j : [0, +\infty) \to M,\ j = m+1, \dots, m+k$$

and

$$u : \Sigma \to T^*M$$

such that $u_j := u \circ \phi_j$ and γ_j are solutions of the system

$$\begin{cases} \frac{d\gamma_j}{dt} = \nabla f_j(\gamma_j) \\ \frac{\partial u_j}{\partial s} + J\frac{\partial u_j}{\partial t} = JX_\rho(u),\ j = 1, \dots, m \\ \frac{\partial u_j}{\partial s} + J\frac{\partial u_j}{\partial t} = JX_{\tilde\rho}(u),\ j = m+1, \dots, k \\ \left[\frac{\partial u}{\partial s} + J\frac{\partial u_j}{\partial t}\right]_{\Sigma_0} = 0 \\ u(\partial\Sigma) \subset 0_M \\ \gamma_j(-\infty) = p_j,\ j = 1, \dots m,\ \gamma_j(+\infty) = p_j,\ j = m+1, \dots m+k \\ \gamma_j(0) = u_j(-\infty, t),\ j = 1, \dots, m, \\ \gamma_j(0) = u_j(+\infty, t),\ j = m+1, \dots, m+k, \end{cases}$$

where ρ is an auxiliary smooth function, equal to 1 on interval $[-R, -2]$ and 0 outside interval $(-R-1, -1)$, and $\tilde\rho(s) = \rho(-s)$). Let $\mathcal{M}_R(\vec{p}, \vec{f}, g; \vec{H}, J)$ be the set of all solutions $(\gamma_1, \gamma_2, \cdots, \gamma_{m+k}, u)$ of this system. The detailed analysis of convergence in $\mathcal{M}_R(\vec{p}, \vec{f}, g; \vec{H}, J)$, based on Arzela–Ascoli theorem, Sobolev theory Gromov compactness is carried out in [6, 7, 9, 22]. The consequence of this analysis is that the topological boundary of every component $\mathcal{M}_R(\vec{p}, \vec{f}, g; \vec{H}, J)$ is the union od lower dimensional components. This conclusion gives rise to definitions of chain homotopy operators that establish the commutativity of the above diagrams (see [12]).

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