The "problem of remainders" in theoretical physics: "physical zeta" function

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Abstract

The present paper is based on the drafts of A.A. Karatsuba's lecture "Physical mathematics in number theory", devoted to the 85th birthday of academician Vasilii Sergeevich Vladimirov. The problem of carefully estimating the errors, which is sometimes missing in the theoretical physics literature is discussed. Possible consequences of such approach when the estimates are missing is demonstrated on example of the Riemann zeta function.

1. Introduction. In many problems of mechanics, quantum physics, physics of wave processes etc. the solutions are represented in the form of the series

$$S = \sum_{a<k\le b} \varphi(k) e^{2\pi i f(k)}. \tag{1}$$

Here $\varphi(x)$ and $f(x)$ are real functions of real argument, $i^2 = -1$; a, b are arbitrary numbers, including $-\infty$ and $+\infty$.

For example, in the relativistic theory of the free fermion Dirac field the commutator function $D(\bar{r}, t)$ is defined by the following expression (see [24], see also [2])

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$$D(\bar{r},t)=\sum_{\bar{k}} L^{-3}e^{i\bar{k}\bar{r}}\;\frac{\sin\left(ct\sqrt{k^2+k_0^2}\right)}{c\sqrt{k^2+k_0^2}}\;, \tag{2}$$

where $\bar{k}$ is the wave vector of the length
$|\bar{k}|=k=\frac{2\pi}{L}\sqrt{k_x^2+k_y^2+k_z^2}$; k_x,k_y,k_z are any integers, $k_0=2\pi\frac{mc}{h}$; m is the mass of the fermion, c is the speed of light, h is the Planck constant; L is the normalizing factor, representing the length of the edge of the cube, in which the eigenfunctions of the operators of the energy-impulse, subject to the periodicity conditions at the walls of this cube, are defined.

The properties of this series are very important in the Dirac theory. It is well-known ([24]) that zeros of the function $D(\bar{r},t)$ correspond to the independent measurements of the fermion charge density in two different points of the space-time. The fermion charge density is one of the few physically observable quantities of quantum field theory.

The series (2) is complicated enough for studying the properties of $D(\bar{r},t)$. That is why physicists, including Dirac ([3], [2], [24], [25]), replaced it by an integral:

$$\widetilde{D}(\bar{r},t)=\frac{1}{(2\pi)^3c}\int_{-\infty}^{+\infty}e^{i\bar{k}\bar{r}}\;\frac{\sin\left(ct\sqrt{k^2+k_0^2}\right)}{c\sqrt{k^2+k_0^2}}\;d\bar{k}\,. \tag{3}$$

However, in [17], [18] is shown that the series (2) and the integral (3) have different properties, in particular, different position of their zeros. So, the length of the time interval containing at least one zero of (2) depends on the relation between the value of the normalizing factor and the Compton wavelength of the fermion. Thus, passing from (2) to (3) we lost certain properties of the original function.

The problem of approximation of the series (1) by a suitable function was studied already by Euler and Poisson.

The number $b-a$ is often called in literature *the length of the sum* S (for the integers a and b, this is the number of the summands in S). Under certain conditions on $\varphi(x)$ and $f(x)$ the sum S can be replaced with a good accuracy by another sum S_1,

$$S_1=\sum_{\alpha<k\le\beta}\Phi(k)e^{2\pi iF(k)},$$

with the length $\beta-\alpha$, which is far smaller than $b-a$. First relations of the form

$$S=S_1+R, \tag{4}$$

with the concrete functions $\varphi(x)$ and $f(x)$, were obtained by Hardy and Littlewood [4], when they derived the approximate functional equation for the Riemann zeta function $\zeta(s)$ and by Vinogradov [30], in the study of the amounts of integer points in the domains on plane. In general form the theorem was proved by Van der Corput [29] (on the recent results connected with it see [22]). With the restrictions on $\varphi(x)$ and $f(x)$, that are convenient for applications, the theorem on the approximation of a trigonometric sum by a shorter one (briefly ATS) was proved by Karatsuba in [6] (see also [13]), the last edition of the ATS with calculated constants is contained in [12]. The application of the ATS in concrete problems of mechanics and quantum optics is developed in [11], [14], [15], [16].

It is natural that the remainder R in (4) can "prevent" the clarification of the behavior and properties of the series S under the study. However the conditions when "influence" of R can be "neglected" should be proved at every concrete case.

In the present paper on the example of a famous series – the Riemann zeta function – we shall study the "physical mathematics" with the replacing the series by the corresponding integral and the results of such a replacement in the theory of prime numbers.

2. "Physical zeta" function. We apply the above-mentioned Dirac's method to the Riemann zeta function $\zeta(s)$. By definition, for $\sigma = \Re s > 1$, $\zeta(s)$ is defined by the following series (see, for example [13]):

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \lim_{N\to+\infty} \sum_{1<n\le N} \frac{1}{n^s} \ . \tag{5}$$

Replacing the last sum by the integral, we find:

$$\sum_{1<n\le N} \frac{1}{n^s} \cong \int_1^N \frac{dx}{x^s} = \frac{1}{1-s} x^{1-s} \Bigg|_1^N = \frac{1}{1-s} N^{1-s} - \frac{1}{1-s} \ .$$

Since $\Re s > 1$, then $N^{1-s} \to 0$ for $N \to +\infty$, and we get:

$$\lim_{N\to+\infty} \sum_{1<n\le N} \frac{1}{n^s} \cong \frac{1}{s-1} \ .$$

From here and from (5) we find:

$$\zeta(s) \cong \frac{s}{s-1} \ . \tag{6}$$

The right hand part of (6) is defined in all s-plane of the complex variable. Therefore, because of the principe of analytic continuation, we have for $\zeta(s)$ the formula (5) for $\Re s > 1$ together with formula (6). The right hand side of (6) is a simplest rational fraction, which has a zero at the point $s = 0$ and a simple pole with residue 1 at the point $s = 1$. Therefore, there is no the Riemann hypothesis, there are no problems with the behavior of $\zeta(s)$, $s = \sigma + it$, $|t| \to +\infty$.

We shall call the derived zeta-function, satisfying the relation (6) by *physical zeta* function and denote by $\widetilde{\zeta}(s)$.

3. "Physical zeta" function in the theory of prime numbers. Let us see what the relation (6) can give in the theory of prime numbers. Note, the function $\zeta(s)$ appears as a function whose properties are closely connected with prime numbers. The function $\zeta(s)$ was introduced in mathematics by Leonard Euler. For $\zeta(s)$ with $\Re s > 1$ the Euler identity is valid:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{7}$$

The identity (7) is the main identity in all applications of $\zeta(s)$ to the theory of prime numbers. Taking the logarithm of (7), we obtain:

$$\ln \zeta(s) = -\sum_p \ln\left(1 - \frac{1}{p^s}\right). \tag{8}$$

Expanding the logarithm in the Taylor series we have:

$$\ln\left(1 - \frac{1}{p^s}\right) = -\frac{1}{p^s} - \frac{1}{2p^{2s}} - \frac{1}{3p^{3s}} - \dots \quad ,$$

$$\ln \zeta(s) = \sum_p \left(\frac{1}{p^s} + \frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + \dots\right) = \sum_p \sum_{k=1}^{\infty} \frac{1}{kp^{ks}}.$$

Finally, differentiating the last equality with respect to s we find

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \sum_{k=1}^{\infty} \frac{\ln p}{p^{ks}} = -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}, \tag{9}$$

where the Mangoldt function $\Lambda(n)$ is introduced,

$$\Lambda(n) = \begin{cases} \ln p, & \text{if} \quad n = p^k, \quad p \text{ is a prime number}, k \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Note, (9) is valid for $\Re s > 1$.

The main functions of the theory of prime numbers are the functions $\pi(x)$ and $\psi(x)$ $(x \geq 2)$:

$$\pi(x) = \sum_{p \leq x} 1, \qquad \psi(x) = \sum_{n \leq x} \Lambda(n).$$

The function $\psi(x)$ is called the Chebyshev function. In spite of its more complicated definition, $\psi(x)$ behaves more simply than $\pi(x)$ when $x \to +\infty$. To be exact, it was proved the statement which is called the law of the distribution of prime numbers (Hadamard and de la Vallée Poussin, 1896) that for $x \to +\infty$

$$\pi(x) \sim Li(x) = \int_2^x \frac{dt}{\ln t}\,, \tag{10}$$

$$\psi(x) \sim x\,. \tag{11}$$

Every from the foregoing relations is equivalent to other one, that is (11) follows from (10) and vice versa, (10) follows from (11).

With the help of a special discontinuous factor it is easy to express $\psi(x)$ via the special integral, whose integrand is equal to the logarithmic derivative of the Riemann zeta function multiplied by $(-x^s/s)$. This special factor $\delta(a)$ is defined by the following integral:

$$\delta(a) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{a^s}{s}\,ds = \begin{cases} 1, & \text{if} \quad a > 1; \\ \frac{1}{2}, & \text{if} \quad a = 1; \\ 0, & \text{if} \quad 0 < a < 1. \end{cases} \tag{12}$$

Here b is an arbitrary number, $b > 0$.

The last relations are proved easily if we translate the line of the integration to the left, if $a > 1$; and to the right, if $0 < a < 1$; and calculate the integral directly, if $a = 1$.

Assume now that x is a noninteger, $x > 1$. Then from (9) and (12) we find for $b > 1$:

$$\psi(x) = \sum_{n=2}^{\infty} \Lambda(n) \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{\left(\frac{x}{n}\right)^s}{s}\,ds = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,ds\,. \tag{13}$$

But if x is an integer, then in the left hand side of (13) we obtain the function $\psi_0(x)$,

$$\psi_0(x) = \frac{\psi(x-0) + \psi(x+0)}{2}\,.$$

The formula (13) is called the Perron formula.

4. "Physical psi-zero" function. Substituting into (13) the "physical zeta" function and taking into account that from (6)

$$-\frac{\widetilde{\zeta}'(s)}{\widetilde{\zeta}(s)} = \frac{1}{s-1} - \frac{1}{s}\,,$$

we get a "physical psi-zero" function which is ($b > 1$):

$$\widetilde{\psi_0}(x) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\left(\frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}\right)x^s\,ds\,.$$

We transfer the line of the integration to the left onto the line $(-Y - i\infty, -Y + i\infty)$, $Y > 1$. We pass in this way two singular points: $s = 1$ and $s = 0$. At these points the integrand has residue which is equal to

$$x - \ln x - 1.$$

Passing to the limit when $Y \to +\infty$, we find:

$$\widetilde{\psi_0}(x) = x - \ln x - 1. \tag{14}$$

5. "Physical pi" function. To derive the "physical pi" function we consider the following sum $F(x)$:

$$F(x) = \sum_{2\le n\le x}\frac{1}{\ln n}\,\Lambda(n)\,. \tag{15}$$

Using the definition of $\Lambda(n)$, we find for $F(x)$ that

$$\begin{aligned} F(x) &= \sum_{p\le x}\frac{1}{\ln p}\ln p + \sum_{p^2\le x}\frac{1}{\ln p^2}\ln p + \sum_{p^3\le x}\frac{1}{\ln p^3}\ln p + \dots \\ &= \pi(x) + \frac{1}{2}\,\pi(\sqrt{x}) + \frac{1}{3}\,\pi(\sqrt[3]{x}) + \dots . \end{aligned}$$

It is clear, that beginning from certain place, all summands of the last sum will be equal to zero, since $\pi(y) = 0$, if $y < 2$. This place is determined by the least natural number k_1, such that $\sqrt[k_1]{x} < 2$, that is, $x < 2^{k_1}$, $k_1 > \frac{\ln x}{\ln 2}$, that is, we can take as a k_1 the number $\left[\frac{\ln x}{\ln 2}\right] + 1$. Thus,

$$F(x) = \sum_{k\ge 1}\frac{1}{k}\,\pi(\sqrt[k]{x}). \tag{16}$$

On the other hand, it is possible to express $F(x)$ via $\psi(x)$. To do this we use the formula of partial summation (Abel's transformation) which has the following form:

$$\sum_{a<n\le b} c(n)f(n) = -\int_a^b C(u)f'(u)\,du + C(b)f(b). \tag{17}$$

Here $c(n)$ are arbitrary complex numbers ,

$$C(u) = \sum_{a<n\leq u} c(n),$$

$f(u)$ is a continuously differentiable function on $[a,b]$. We apply (17) to (15), setting $a = 2-\varepsilon$, $\varepsilon \to 0$, $b = x$, $f(n) = \frac{1}{\ln n}$, $c(n) = \Lambda(n)$. Then we get:

$$C(u) = \sum_{2\leq n\leq u} \Lambda(n) = \psi(u), \qquad f'(u) = -\frac{1}{u\ln^2 u}\ ,$$

$$F(x) = \int_2^x \frac{\psi(u)\,du}{u\ln^2 u} + \frac{\psi(x)}{\ln x}\ . \tag{18}$$

We substitute (14) into (18) (note that $\psi_0(u)$ is different from $\psi(u)$ only in a finite number of points of the interval $[2,x]$) and find:

$$\begin{aligned}\widetilde{F}(x) &= \int_2^x \frac{u-\ln u-1}{u\ln^2 u}\,du + \frac{x-\ln x-1}{\ln x}\\ &= -\int_2^x u\,d\frac{1}{\ln u} - \int_2^x \frac{d\ln u}{\ln u} - \int_2^x \frac{d\ln u}{\ln^2 u} + \frac{x}{\ln x} - 1 - \frac{1}{\ln x}\\ &= Li(x) - \ln\ln x + \frac{1}{\ln 2} + \ln\ln 2 - 1.\end{aligned}$$

Thus, we obtain the next "physical" formula:

$$\widetilde{F}(x) = Li(x) - \ln\ln x + c, \tag{19}$$

where

$$c = \frac{1}{\ln 2} + \ln\ln 2 - 1.$$

Now we come back to (16) and rewrite it in the following form:
for $x \geq 2$,

$$\sum_{k\leq 2\ln x} \frac{1}{k}\,\pi(x^{\frac{1}{k}}) = F(x).$$

Then the following equality holds:

$$\pi(x) = \sum_{k\leq 2\ln x} \frac{1}{k}\,\mu(k)\,F(x^{\frac{1}{k}}), \tag{20}$$

where $\mu(k)$ is the Möbius function,

$$\mu(k) = \begin{cases} 1, & \text{if} \quad k = 1 \\ (-1)^\nu, & \text{if} \quad k = p_1 \dots p_\nu; \quad p_j \ \text{are different prime numbers} \\ 0, & \text{if} \quad k = p^2 m; \quad p \ \text{is a prime number.} \end{cases}$$

Indeed, we have by the definition:

$$F(x^{\frac{1}{k}}) = \sum_{l \le 2 \ln x^{1/k}} \frac{1}{l}\, \pi(x^{\frac{1}{kl}}) = \sum_{l \le \frac{2}{k} \ln x} \frac{1}{l}\, \pi(x^{\frac{1}{k\,l}})\,.$$

Substituting the last expression into right hand side of (20) we find:

$$\sum_{k \le 2 \ln x} \frac{1}{k}\, \mu(k) \sum_{l \le \frac{2}{k} \ln x} \frac{1}{l}\, \pi(x^{\frac{1}{kl}}) = \sum_{kl \le 2 \ln x} \frac{1}{k\,l}\, \pi(x^{\frac{1}{k\,l}}) \mu(k)$$

$$= \sum_{n \le 2 \ln x} \frac{1}{n}\, \pi(x^{\frac{1}{n}}) \sum_{l\,k = n} \mu(k) = \sum_{n \le 2 \ln x} \frac{1}{n}\, \pi(x^{\frac{1}{n}}) \sum_{k | n} \mu(k) = \pi(x)\,.$$

We use here the well-known property of the Möbius function:

$$\sum_{k|n} \mu(k) = \begin{cases} 1, & n = 1, \\ 0, & n > 1, \end{cases}$$

(the sign $k|n$ denotes that k divides n). To obtain a "physical" $\pi(x)$ we substitute into (20) the value of $F(x)$ from (19):

$$\widetilde{\pi}(x) = \sum_{k \le 2 \ln x} \frac{1}{k}\, \mu(k) \left(Li \left(x^{\frac{1}{k}} \right) - \ln \ln x^{\frac{1}{k}} + c \right)$$

$$= \sum_{k \le 2 \ln x} \frac{\mu(k)}{k} Li \left(x^{\frac{1}{k}} \right) - \sum_{k \le 2 \ln x} \frac{1}{k}\, \mu(k) \left(\ln \ln x^{\frac{1}{k}} + c \right).$$

The second sum in the last relation is very small, it doesn't exceed by absolute value the value

$$\sum_{k \le 2 \ln x} \frac{1}{k} \left(\ln \ln x + |c| \right) \le \left(\ln \ln x + |c| \right) \left(\ln \ln x + 2 \right).$$

Therefore, we find the following "physical" asymptotic formula:

$$\widetilde{\pi}(x) = \sum_{k \ge 1} \frac{\mu(k)}{k} Li \left(x^{\frac{1}{k}} \right) + \theta (\ln \ln x + c_1)^2, \tag{21}$$

where c_1 is a constant, $|\theta| \leq 1$, $Li(y) = \int_2^y \frac{dt}{\ln t}$ for $y \geq 2$, and $Li(y) = 0$ for $y < 2$. The expanded form of (21) looks like

$$\widetilde{\pi}(x) = Li(x) - \frac{1}{2}Li(\sqrt{x}) - \frac{1}{3}Li(\sqrt[3]{x}) - \frac{1}{5}Li(\sqrt[5]{x}) + \frac{1}{6}Li(\sqrt[6]{x}) - \frac{1}{7}Li(\sqrt[7]{x}) + \frac{1}{10}Li(\sqrt[10]{x}) - \cdots + .$$

We denote by $B(x)$ (the letter B is taken in honour of Bernhard Riemann) the sum in k in right hand side of (21), that is,

$$B(x) = \sum_{k \geq 1} \frac{\mu(k)}{k}\, Li\left(x^{\frac{1}{k}}\right). \tag{22}$$

Thus, the "physical" formula for the pi-function shows that the pi-function is very well approximated by the function $B(x)$.

We denote by $R(x)$ the following difference:

$$R(x) = \pi(x) - Li(x). \tag{23}$$

6. Problem of approximation of the function $\pi(x)$. The problem of the behavior of the function $\pi(x)$ and of the approximation of $\pi(x)$ by "good" functions is one of the most old and interesting problem in the theory of numbers.

Euclid (3 century B.C.E.) proved that

$$\pi(x) \to +\infty, \qquad x \to +\infty.$$

Legendre (1798) presupposed that

$$\pi(x) = \frac{x}{A \ln x + B},$$

then (1808) that

$$\pi(x) = \frac{x}{\ln x + B}, \qquad B \to -1.08366\ldots.$$

Chebyshev (1848) disproved the Legendre hypothesis and proved the estimates:

$$\underline{\lim} \frac{\pi(x)}{Li(x)} \leq 1 \leq \overline{\lim} \frac{\pi(x)}{Li(x)}.$$

Chebyshev proposed the hypothesis that the best approximation for $\pi(x)$ can be only $Li(x)$.

The same hypothesis was formulated independently by Gauss in 1849 (in a letter which was published in 1863).

In his famous manuscript (1859, [23]) Riemann wrote that the better approximation to $\pi(x)$ is sooner the function $B(x)$. As he noted, this fact is confirmed by calculations performed by Gauss and Goldschmidt up to $x \leq 3000000$. Next more evident confirmation of this proposition is the recent calculations performed with the help of modern computers ([19], [27]), a small part of which is presented below in the following table:

x	$\pi(x)$	$B(x) - \pi(x)$	$R(x)$
10^8	5761455	97	−754
10^9	50847534	−79	−1701
10^{10}	455052511	−1828	−3104
10^{11}	4118054813	−2318	−11588
10^{12}	37607912018	−1476	−38263
10^{13}	346065536839	−5773	−108971
10^{14}	3204941750802	−19200	−314890
10^{15}	29844570422669	73218	−1052619
...	...	...	...
10^{22}	201467286689315906290	−127132665	−1932355208

Certain corollaries from the "physical" formulas and its comparison with mathematically correct results in the theory of prime numbers. As Riemann noted, in the limits of calculations (see table) the function $R(x)$ is always negative, and $|R(x)|$ increases when x increases. The "physical" formula (21) confirms this note of Riemann. At the same time, Littlewood (1914, [21]) proved that $R(x)$ changes sign infinitely often. Moreover, for $R(x)$ the following relation is valid:

$$R(x) = \Omega_{\pm}\left(\sqrt{x}\,\frac{\ln\ln\ln x}{\ln x}\right),$$

(the equality $f(x) = \Omega_{\pm}(g(x))$ means that there are two sequences $\{u_j\}$, $\{v_j\}$, tending to infinity and such that $f(u_j) > cg(u_j)$, $\quad f(v_j) < -cg(v_j)$, $j = 1, 2, \ldots$, where c is a constant).
Therefore, the "physical" formula for $\pi(x)$ gives the incorrect result. This incorrect result was obtained because of the reason that $\zeta(s)$ was incorrectly continued onto all s-plane. The correct continuation is much more complicated:

$$\zeta(s) = \frac{1}{s-1}\, e^{As+B} \left(\prod_{n=1}^{\infty}\left(1 - \frac{s}{\rho_n}\right) e^{\frac{s}{\rho_n}}\right) \left(\prod_{n=1}^{\infty}\left(1 + \frac{s}{2n}\right) e^{-\frac{s}{2n}}\right).$$

Here A and B are the constants,

$$A = 1 - 2\int_1^{\infty} \frac{\sigma(x)\,dx}{x^2}\,, \qquad\qquad B = -\ln 2,$$

$$\sigma(x) = \int_0^x \varrho(u)\,du, \qquad \varrho(u) = \frac{1}{2} - \{u\},$$

and for a real x, the function $y = \{x\}$ is the fractional part of the number x, that is $y = \{x\} = x - [x]$, where $[x]$ is the integral part of x, that is an integer such that $[x] \le x < [x] + 1$;

ρ_n is a discrete sequence of complex numbers of the form $\rho_n = \beta_n + i\,\gamma_n$, where $0 \le \beta_n \le 1$, and $|\gamma_n| > 0$, $n = 1, 2, \ldots$.

The numbers ρ_n are nontrivial zeros of $\zeta(s)$. The Riemann hypothesis claims that $\beta_n = \frac{1}{2}$.

The numbers $-2, -4, -6, \ldots, -2n, \ldots$ are trivial zeros of $\zeta(s)$.

There are also other formulas continuing $\zeta(s)$:

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + s\int_1^\infty \frac{\rho(u)\,du}{u^{s+1}} = \frac{1}{2} + \frac{1}{s-1} + s(s+1)\int_1^\infty \frac{\sigma(u)\,du}{u^{s+2}}\,.$$

The difference between the functions $\widetilde{\zeta}(s)$ and $\zeta(s)$ is distinct. It is clear that logarithmic derivative of $\zeta(s)$ will be much more complicated of the "physical" one, also as the corresponding formulas for $\psi(x)$ and $\pi(x)$.

The right formula for $\psi_0(x)\,(x > 1)$:

$$\psi_0(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\ln\left(1 - \frac{1}{x^2}\right), \tag{24}$$

where

$$\sum_\rho \frac{x^\rho}{\rho} = \lim_{Y\to+\infty} \sum_{|\gamma|\le Y} \frac{x^\rho}{\rho}$$

differs very much from "physical psi-zero" (14). It is obvious that right form of the function $\pi(x)$ obtained from (24) with the help of the formula of Möbius inversion will be much more complicated than "physical pi" function.

7. Function $R(x)$ and the approximation of $\pi(x)$ by the function $B(x)$. We formulate briefly the results on the behavior of $R(x)$ and on the approximation of $\pi(x)$ by $B(x)$.

We denote

$$r(x) = \pi(x) - B(x).$$

Remind that Littlewood proved that $R(x)$ changes sign infinitely often, besides,

$$R(x) = \Omega_{\pm}\left(\sqrt{x}\,\frac{\ln\ln\ln x}{\ln x}\right),$$

and in addition the relation

$$R(x) = O\left(\sqrt{x}\ln x\right)$$

is equivalent to the Riemann hypothesis (RH).

Assuming RH, Ingham proved (1932, [5]) that

$$-\,0.72\,\frac{x^{\frac{3}{2}}}{\ln x} < \int_2^x R(u)\,du < -\,0.62\,\frac{x^{\frac{3}{2}}}{\ln x},$$

that is $R(u)$ "on the average" is a negative one, and $|R(u)| \to +\infty$ when $u \to +\infty$.

Karatsuba proved (2004, [7], [8]) the following statement:

Theorem 1 *Assume that for $j = 1, 2, 3, \ldots$ the integrals $I_j(x)$ are defined by the formulas*

$$I_1(x) = \int_2^x r(t)\,dt, \qquad I_2(x) = \int_2^x I_1(t)\,dt, \quad \ldots \quad I_{j+1}(x) = \int_2^x I_j(t)\,dt.$$

Then for every j there exist the numbers $X_j \geq 2$, $a_j \geq 2$, such that for $X \geq X_j$ on every interval of the form $(X, a_j X)$ the function $I_j(x)$ changes sign.

At first Karatsuba proved the theorem 1 assuming RH. In 2005 Karatsuba proved (see [9], [10]) this theorem absolutely, that is, without the assumption of RH.

It follows from the theorem 1 that for $X \geq X_j$ the function $I_j(x)$ changes sign on $[2, X]$ at least $c_j \ln X$ (c_j are the constants) times.

The t Theorem 1 confirms the fact that the Riemann approximation of the function $\pi(x)$ by the function $B(x)$ is more useful than the approximation of $\pi(x)$ by the function $Li(x)$ ("Gauss" approximation).

8. "Physical zeta" function is adequate to the "physical world" of small numbers. There is the following remarkable fact, the Littlewood result on the changing the sign of the function $R(x)$ from (23), to what "physical pi" function doesn't satisfy, is valid on very long intervals.

Skewes (1933, [26]) proved that $R(x)$ changes sign on the interval $(2, x_1)$, where

$$x_1 < 10^{10^{10^{34}}}. \tag{25}$$

Hardy called the number in the right hand side of (25) as a "largest number encountered in mathematical proofs". In 1966 (see [20]) Lehman proved that

$$x_1 < 1.65 * 10^{1165},$$

and Te Riele ([28]) proved in 1987 that

$$x_1 < 7 * 10^{370}.$$

Recently Bays and Hudson (2000, [1]) proved the estimate

$$x_1 < 1.39822 * 10^{316}. \tag{26}$$

The foregoing numbers are very large, there are no such physical values. By recent estimations (CODATA-2002), the mass of Universe is equal $2.25174 * 10^{50}$ kg, conventional number of hydrogen atoms in Universe is $1.34550 * 10^{77}$, volume of Universe is equal $1.59920 * 10^{77}$ m^3; from the smallest physical constants are the electron mass $9.10938 * 10^{-31}$ kg, the proton mass $1.67262 * 10^{-27}$ kg, the Compton wavelength of the electron $2.42531 * 10^{-12}$ m, classical electron radius $2.81794 * 10^{-15}$ m.

In the universally recognized Friedman model, our Universe is a finite object expanding with time. From the restrictions on the limits of possible measurements of physical values it follows that "physical world" consists of a finite total number of components of a minimal measurable size. Even if the experiments on the new Large Hadron Collider (CERN, built to 2008) possibly increase this total number, this increasing doesn't change the picture in principle, the number of the physical "indivisible" components of a "minimal size" will be far from the number (26). In such a "physical world" the "physical zeta" function works well and gives correct results in the theory of prime numbers.

Mathematics operates with very large and very small numbers, roughly speaking, it operates with infinity and with zero.

It is possible to imagine two absolutely different ways of our Universe expansion.

The first way assumes the increasing of the largest possible size of a physical quantity to the infinity and the decreasing the smallest possible size to zero. So that with time, the finite "physical world" will approximate the infinite "mathematical world", and from certain moment "physical formulas" will not provide adequate results. However, in this case, it seems (at least from the restrictions in quantum theory) that with such expansion of Universe certain "absolute" physical constants must change their values (for example, the speed of light may increase etc.), that is, they are not absolute.

Another way describes the well-known model of the Universe, expanding and contracting with time, based on Friedman's solutions. If the maximal possible size of such an oscillating Universe assumes participation of not very large numbers, then "physical zeta" function will be an adequate function to this "world" and many other "physical formulas" will provide also adequate results. Thus the finiteness and relative smallness of the total number of components of "physical world" supports the "physical mathematics."

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