Analytic Continuation by Duality using the Approximation with an Asymmetric Weighting Function

S. R. Ignjatović*
Department of Physics, Prirodno-matematički fakultet,
Mladena Stojanovića 2, 78000 Banja Luka,
BOSNIA AND HERZEGOVINA

Abstract

The Analytic Continuation by Duality (ACD) method is used to estimate dispersive integrals in low-energy QCD phenomenology and in technicolor models. The method uses a polynomial approximation of the $1/x$ function, usually a best-L_p approximation. There are several sources of error in the ACD estimates which, along with the oscillatory behavior of the best-Lp approximations, render them unreliable. In an attempt to improve the ACD estimates, we use the weighted least-squares approximations. A weight function that emphasizes the better-known large-momentum expansion of the vacuum polarization near the high-energy (ultra-violet) end of the integral is expected to improve the estimates. We have checked such approximations on a simple model spectrum - two infinitely sharp resonances - since for this model the fitting (polynomial approximation) is the only source of error. The weighted approximations yield less reliable estimates than the unweighted approximation. Therefore, another method of approximation is needed to minimize the fitting error in the ACD method, although this may require some information about the spectral function.

1. Introduction

Shifman, Vainshtein and Zakharov extended the QCD to the subasymptotic region [1] using the Operator Product Expansion (OPE) of the current-current correlators. They obtained theoretical predictions of many low-energy QCD effects, although the vacuum condensates – which figure in the OPE coefficients – were taken as an input.

The Analytic Continuation by Duality (ACD) is used to evaluate dispersive integrals in QCD using the first few terms of the OPE. It is claimed that no

* e-mail address: sinisha@teol.net

other phenomenological input is needed. The ACD method was developed in [2] and applied to several QCD effects and in technicolor models [3]. The ACD approximates the kernel of a dispersive integral by a polynomial on the real line, which allows the integration to be performed in the complex plane.

The analytic continuation is an ill-posed problem: small variations in the input parameters may lead to large variations in the results. We have discussed this problem for the ACD in our previous contribution [4], where we have also related the instability of the method to the oscillatory character of the best-L_p polynomial approximations. In the modified version of the ACD method [5], the $1/s$ kernel is approximated by truncated Taylor series around the upper cut-off R. Supposedly, the modified method gives more reliable estimates because the polynomial approximation is better near the upper cut-off, where the OPE is more accurate. This method is stable, but it yields only a lower bound for the esimates [4, 6].

The reliability of the ACD-derived estimates has been investigated for the original ACD [7], which we found to be unreliable due to several sources of error that are difficult to control. An instability was found for most models and divergence for some models. Here we investigate the reliability of the ACD for a simple model spectrum – vector-meson dominance with two infinitely sharp resonances – and with several weighted best-L_2 approximations. A weight function that emphasizes the better-known large-momentum expansion of the vacuum polarization near the high-energy end of the integral is supposed to improve the estimates.

The unknown spectrum of the Walking Technicolor is particularly sensitive to the reliability problems of the ACD and this was the original motivation behind our investigation [3, 7]. In this contribution, we do not refer to any physical problems as the ACD is applicable to diverse phenomena in high- and intermediate-energy physics.

2. ACD Method

Finding the matrix element of a process with the vacuum polarization $\Pi(s)$ in the non-perturbative regime may lead to the dispersive integral

$$\int_{s_0}^{\infty} \frac{1}{s}\,\mathrm{Im}F(s)ds,$$

$F(s)$ being a simple function of $\Pi(s)$ and s_0 the threshold of the process.

The function $F(s)$ is analytic in the entire complex plane except on the branch cut along the real axis from s_0 so the Cauchy's integral formula gives

$$F(t) = \frac{1}{2\pi i}\oint_C \frac{F(s)}{s-t}ds, \qquad (1)$$

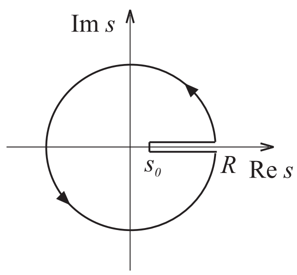

Figure 1: The contour of integration.

where C is the contour shown in Fig. 1. The integral consists of two parts [4]:

$$F(t) = \frac{1}{2\pi i} \oint_{|s|=R} \frac{F(s)}{s-t} ds + \frac{1}{\pi} \int_{s_0}^{R} \frac{\mathrm{Im}F(s)}{s-t-i\varepsilon} ds. \tag{2}$$

In the limit $t \to 0$ Eq. 2 becomes

$$F(0) = \frac{1}{2\pi i} \oint_{|s|=R} \frac{F(s)}{s} ds + \frac{1}{\pi} \int_{s_0}^{R} \frac{\mathrm{Im}F(s)}{s} ds. \tag{3}$$

It can be shown that asymptotically $F(s) \sim 1/s$ so the integral over the circle $|s| = R$ vanishes as $R \to +\infty$. Therefore, in this limit we get the dispersion relation

$$\overline{F} \equiv F(0) = \frac{1}{\pi} \int_{s_0}^{+\infty} \frac{\mathrm{Im}F(s)}{s} ds. \tag{4}$$

In contrast to the approach that leads to the dispersion relation (4), the ACD method makes the *second* integral in (3) vanish. Approximating the kernel $1/s$ by the polynomial

$$p_N(s) = \sum_{n=0}^{N} a_n(N) s^n$$

we find [4]

$$F(0) = \frac{1}{2\pi i} \oint_{|s|=R} \left[\frac{1}{s} - p_N(s)\right] F(s)\, ds + \frac{1}{\pi} \int_{s_0}^{R} \left[\frac{1}{s} - p_N(s)\right] \mathrm{Im}F(s)\, ds. \tag{5}$$

Eq. (5) is exact. If we find $p_N(s)$ that converges to $1/s$ on the interval $[s_0, R]$ then the second integral in (5) will tend to be small and will be called "fit error":

$$\delta F_{\mathrm{fit}}(N) = \frac{1}{\pi} \int_{s_0}^{R} \left[\frac{1}{s} - p_N(s)\right] \mathrm{Im}F(s)\, ds. \tag{6}$$

The fit error depends not only on N but also on 1) the interval $[s_0, R]$ and 2) the fit routine. The fit routines will be considered in Sec. 3.

Let us assume that the large-s expansion

$$F(s) = \sum_{m=1}^{M} \frac{h_m(s)}{s^m} + O\left(\frac{1}{s^{M+1}}\right) \tag{7}$$

is valid around the circle $|s| = R$. The expansion (7) may be OPE, which is an asymptotic series, or the large-momentum expansion of an exactly known vacuum polarization, which may be convergent series (as is the case for the model considered in Sec. 4).

The truncation of the series (7) introduces a further error and the neglect of the momentum dependence of the large-momentum expansion coefficients h_m an additional error [4]. We do not consider these two errors here since the fit error is the only error for our model. The only non-zero contributions to the first integral in (3) come from the simple poles so that

$$F_{\mathrm{ACD}} = -\sum_{n=0}^{\min(N,M-1)} \hat{h}_{n+1} a_n(N). \tag{8}$$

If the spectral function is known exactly, the dispersion relation gives a reliable estimate of $\overline{F}$ so the comparison to F_{ACD} will determine if the ACD estimate is reliable.

3. Best-L_2 Polynomial Approximations of the Function $1/s$

We need to approximate the function $1/s$ by the polynomial

$$p_N(s) = \sum_{n=0}^{N} a_n s^n \tag{9}$$

in the interval $[s_0, R]$, where the coefficients a_n depend on N. With a change of variables $x = -ps + q$, where

$$p = \frac{2}{R - s_0}, \qquad q = \frac{R + s_0}{R - s_0}$$

the problem is reduced to fitting the function

$$f(x) = \frac{1}{x - q}, \qquad q > 1 \tag{10}$$

with the polynomial

$$\tilde{p}_N(x) = \sum_{k=0}^{N} b_k x^k \tag{11}$$

in the interval $[-1, 1]$. Note that the change of variables has rescaled and *inverted* the interval: s_0 corresponds to $+1$ and R to -1. There is also an overall factor $-p$:

$$\frac{1}{s} = \frac{-p}{x - q}.$$

Therefore,

$$p_N(s) = -p\, \tilde{p}_N(x). \tag{12}$$

Let us express a_n through b_n. From (9), (11) and (12) we get

$$\sum_{n=0}^{N} a_n s^n = -p\sum_{k=0}^{N} b_k(-ps+q)^k = \sum_{k=0}^{N}\sum_{i=0}^{k} b_k \binom{k}{i}(-p)^{i+1}q^{k-i}s^i,$$

which gives

$$a_n = \sum_{k=n}^{N} \binom{k}{i}(-p)^{n+1}q^{k-n}b_k. \tag{13}$$

Thus the problem is reduced to finding the coefficients b_k. Consider the polynomials $y_n(x)$ that minimize the functional distance

$$\Delta_N = \int_a^b \left[f(x) - \sum_{n=0}^{N} c_n y_n(x)\right]^2 \rho(x)dx, \tag{14}$$

where $f(x)$ and $\rho(x)$ are functions that are continuous in $[a, b]$, i. e. the best-L_2 (least-squares) approximation. The polynomials $y_n(x)$ obey the orthonormality relation

$$\int_a^b y_n(x)y_m(x)\rho(x)dx = d_n^2\delta_{mn},$$

where d_n is a constant that depends on the kind of polynomials. It can be show that the minimization of Δ_N such that $\lim\limits_{N\to+\infty}\Delta_N = 0$ is achieved if c_n are the Fourier coefficients of $f(x)$ in the $\{y_n(x)\}$ basis [8]:

$$c_n = \frac{1}{d_n^2}\int_a^b f(x)y_n(x)\rho(x)dx. \tag{15}$$

Any finite interval $[a, b]$ can be scaled to the interval $[-1, 1]$ by a simple change of variables, just as we did for the interval $[s_0, R]$. Specifying the formula to the function (10) and to the interval $[-1, 1]$ we find

$$c_n = \frac{1}{d_n^2}\int_{-1}^{1} \frac{y_n(x)\rho(x)}{x-q}dx, \qquad q > 1. \tag{16}$$

The classical orthogonal polynomials that correspond to this interval are the Jacobi polynomials, whose weight function is

$$\rho(x) = (1-x)^\alpha(1+x)^\beta, \qquad \alpha > -1, \quad \beta > -1. \tag{17}$$

On the other hand, the integral on the right-hand side of (16) for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ can be expressed in terms of the Jacobi functions of the second kind [9]:

$$Q_n^{(\alpha,\beta)}(q) = \frac{1}{2(q-1)^\alpha(q+1)^\beta}\int_{-1}^{1} \frac{P_n^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta}{q-x}dx. \tag{18}$$

Comparing (16) with $\rho(x)$ given in (17) with (18) we find

$$c_n^{(\alpha,\beta)}(q) = -\frac{Q_n^{(\alpha,\beta)}(q)}{d_n^2(\alpha,\beta)} 2(q-1)^\alpha (q+1)^\beta. \tag{19}$$

In principle, α and β are free parameters. However, the requirements of an effective application of the ACD method constrain the parameters considerably. The unweighted best-L_2 approximation ($\rho = 1$) corresponds to Legendre polynomials $P_n^{(0,0)}(x) = P_n(x)$. The weight function with $\alpha = \beta = -1/2$, which corresponds to Chebyshev polynomials of the first kind, improves the approximation towards the ends of the interval. This approximation will tend to approach the best-L_∞ approximation, which we have shown to yield highly oscillatory estimates [7]. The weight function with $\alpha = \beta = 1/2$ corresponds to Chebyshev polynomials of the second kind and improves the approximation towards the middle of the interval. This approximation will tend to approach the best-L_1 approximation, which yields less oscillatory estimates [7].

The OPE is more reliable near the upper bound R of the interval, which corresponds to $x = -1$; it approximates $F(s)$ better with a given number of terms. Hence we expect that improving the polynomial approximation near $x = -1$ will tend to improve the reliability of the estimates. An attempt in this direction was made by expanding $1/s$ into Taylor series around $s = R$ [5], but this yields only a lower bound for the ACD estimates [6]. According to this reasoning, the reliability of the estimates should be improved with best-L_2 approximation weighted towards R, i. e. $x = -1$. Since $\beta > -1$ a natural choice is $\beta = -1/2$ with $\alpha \geq 0$.

Weighting away from the other end ($x = -1$) is expected to yield less oscillatory behavior so we choose $\alpha = 1/2$. Thus the weight function is

$$\rho(x) = \sqrt{\frac{1-x}{1+x}}. \tag{20}$$

The weight function (20) is also computationally convenient since the corresponding polynomials – the so-called Chebyshev polynomials of the third kind $V_n(x)$ [10] – have certain properties. The Chebyshev polynomials of the third and fourth kind can be expressed as

$$V_n(\cos\theta) = \frac{\sin\left[\left(n+\frac{1}{2}\right)\arccos x\right]}{\sin\left(\frac{1}{2}\arccos x\right)}, \quad W_n(\cos\theta) = \frac{\cos\left[\left(n+\frac{1}{2}\right)\arccos x\right]}{\cos\left(\frac{1}{2}\arccos x\right)},$$

where $\cos\theta = x$ (there is also an opposite notation in which V_n and W_n are interchanged). The polynomials V_n and W_n are called 'airfoil' polynomials u_n ant t_n, respectively, in engineering mathematics, where they have been extensively studied [11].

What makes the polynomials V_n and W_n particularly well-suited for approximation of the function (10) is their property

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,\frac{V_n(x)}{x-q}dx = -W_n(q), \quad \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,\frac{W_n(x)}{x-q}dx = V_n(q), \tag{21}$$

where Cauchy's principal value of the integrals is understood. The normalization constants are $d_n^2 = \pi$ for both V_n and W_n. Therefore, relations (16), (20) and (21) give

$$c_n^{III} = -W_n(q), \qquad c_n^{IV} = V_n(q), \tag{22}$$

where the upper index refers to the kind of Chebyshev polynomials used in the approximation. The polynomials

$$\tilde{p}_N(x) = \sum_{n=0}^{N} c_n^{III} V_n(x), \qquad \tilde{p}_N(x) = \sum_{n=0}^{N} c_n^{IV} W_n(x)$$

need to be converted into power series (11), which can be done using recursively evaluated (or tabulated [11]) V_n and W_n. Then a further conversion to a_n is done from (13).

For comparison purposes, we also consider approximations by Chebyshev polynomials of the first and second kinds as well as by Legendre polynomials. For Chebyshev polynomials of the first and second kinds relations similar to (21) hold [12]:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,\frac{T_n(x)}{x-q}dx = U_{n-1}(q), \quad \frac{1}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\,\frac{U_{n-1}(x)}{x-q}dx = -T_n(q). \tag{23}$$

The integral of $T_0 = 1$ needs to be evaluated separately:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,\frac{1}{x-q}dx = -\frac{1}{\sqrt{q^2-1}} = c_0(q). \tag{24}$$

The normalization constants are $d_n^2 = \pi/2$, except $d_n^2 = \pi$ for $T_0 = 1$. Hence from (16) with appropriate weight functions and (23) we get

$$c_n^{I} = 2U_{n-1}(q), \qquad c_n^{II} = -2T_{n+1}(q), \tag{25}$$

and using (24) $c_0^I = c_0(q)$.

For Legendre polynomials $d_n^2 = 2/(2n+1)$ so Eq. (19) gives

$$c_n^0 = -(2n+1)Q_n(q), \tag{26}$$

where $Q(q)$ are Legendre functions of the second kind. From (18) with $\alpha = \beta = 0$ we get

$$Q_0(q) = \frac{1}{2}\ln\frac{q+1}{q-1}, \qquad Q_1(q) = q\,Q_0(q) - 1$$

and the other functions can be found from Bonnet's recursive formula:

$$(n+1)Q_{n+1}(q) = (2n+1)q\,Q_n(q) - nQ_{n-1}(q)$$

since the functions of the second kind obey the same recursive relations.

The only other weight function besides (20) that appears promising is

$$\rho(x) = \frac{1}{\sqrt{1+x}}, \tag{27}$$

although the polynomials $P_n^{(0,-1/2)}(x)$ are computationally inconvenient. We have obtained some estimates with this weight as well.

4. ACD Estimates for Vector-Meson Dominance Model

In order to investigate the stability of the ACD-derived results we introduce a simple model function $F(s)$ for which the dispersive integral (4) can be evaluated exactly. In the model, there is a single resonance of mass m_V in the vector channel and a single resonance of mass m_A in the axial channel. The parameters of the resonances are constrained by the condition that the spectra saturate the first and second Weinberg sum rules.

The first model contains two infinitely sharp (δ-function) resonances:

$$\mathrm{Im}F(s) = -\pi[f_V^2\delta(s-m_V^2) - f_A^2\delta(s-m_A^2)],$$

$$F(s) = \frac{f_V^2}{s-m_V^2+i\epsilon} - \frac{f_A^2}{s-m_A^2+i\epsilon}.$$

Clearly,

$$\overline{F} = F(0) = \frac{f_A^2}{m_A^2} - \frac{f_V^2}{m_V^2} \tag{28}$$

provided $s_0 < m_V, m_A < R$. The Weinberg sum rules give [4]

$$\overline{F} = (1+r)\frac{f^2}{m_V^2}, \tag{29}$$

where $r = m_V^2/m_A^2$ (for QCD $r \approx 0.4$). The large-momentum expansion is

$$F(s) = \sum_{n=0}^{+\infty}\frac{f_V^2 m_V^{2n} - f_A^2 m_A^{2n}}{s^{n+1}} = \frac{f^2}{s} + \sum_{n=2}^{+\infty}\frac{f^2 m_V^{2n}}{s^{n+1}}\frac{1-r^{1-n}}{1-r}.$$

In general, only a few terms of the OPE are known [1, 2] so in actual applications $M \leq 5$, while N is under our control. We have adopted the logical choice $N = M - 1$. In order to check the sensitivity of the estimates to the location of the resonances we have calculated F_{ACD} not only for its QCD value $r = 0.4$, but also for $r = 0.3$ and $r = 0.5$. This is equivalent to varying the mass m_A of the axial channel resonance with the mass m_V of the vector channel resonance fixed. The results are shown in Table 1.

We can see that all the weighted routines yield highly unstable and unreliable results. The results for V_n-fits (Chebyshev polynomials of the third kind) are somewhat more stable than for other weighted routines, but still far worse than for the unweighted fit. We have included the 'wrongly' weighted W_n-fit as a check up. We have not checked the possibility of milder weighting ($\beta \sim -0.2$, say) as this would have only given a moderate improvement over the unweighted P_n-fit at the expense of introducing another free parameter.

For the weight function (27) we have calculated only 3 values of F_{ACD}, which are, in units of its exact value, 0.324, 0.048, −0.057 for $N = 0, 1, 2$, respectively. Hence this routine is worse than V_n-fit.

The dependence of the estimates on the cutoffs is shown in Table 2. We have included only P_n- and V_n-fits since the other routines are essentially excluded by the rather poor results shown in Table 1. As expected, the weight function of the V_n-fit damps the s_0-dependence while enhancing the R-dependence. The strong s_0-dependence is a known problem of the ACD method [13], but the R-dependence may be even less tractable since there is no well-defined threshold of the validity of the subasymptotic QCD.

The instability of all results as N is increased beyond unity is not peculiar to the weighted routines, but it is common to all the best-L_p routines. The reason is that the difference $\Delta_N = 1/s - p_N(s)$ starts to oscillate within the resonance region of the spectrum so when extrema of the spectral function $\mathrm{Im}F(s)$ happen to be near the extrema of Δ_N the fit error (6) is strongly enhanced.

5. Conclusions

We have checked if the ACD-derived estimates can be improved by weighting the polynomial fits to $1/s$ towards larger s where the large-s expansion is more trustworthy. For the simple vector-meson dominance model, the results are much worse than for the unweighted best-L_2 routine. This suggests that the ACD method – for realistic order of large-s expansion – relies on minimization of the fit error itself, which includes the unknown spectral function $\mathrm{Im}F(s)$, rather than on the accuracy of the large-s expansion.

The oscillations of the fit polynomial of order $N > 1$ within the resonance region (i. e. near the extrema of $\mathrm{Im}F(s)$) make the estimates unreliable for the physically most interesting cases $N = 2, 3$. Thus knowing the higher-

Table 1: F_{ACD} in units of its exact value (29) for 3 values of the dimensionless parameter r. The cutoffs are $s_0 = 0.2m_V^2$ and $R = 5m_V^2$.

r	N	P_n-fit	T_n-fit	U_n-fit	V_n-fit	W_n-fit
0.3	0	0.516	0.769	0.694	0.321	−0.321
	1	1.290	0.074	2.368	1.504	−1.504
	2	1.544	−0.189	4.189	3.241	−0.026
	3	1.263	−1.965	2.757	3.606	6.784
	4	1.008	9.144	−1.409	1.077	61.049
0.4	0	0.479	0.714	0.645	0.298	−0.298
	1	1.198	0.069	2.199	1.397	−1.397
	2	1.564	−0.549	4.522	3.273	−1.034
	3	1.375	−1.467	3.922	4.278	5.609
	4	1.018	10.188	−2.072	1.504	43.702
0.5	0	0.447	0.666	0.602	0.278	−0.278
	1	1.118	0.064	2.052	1.304	−1.304
	2	1.533	−0.721	4.575	3.202	−1.530
	3	1.450	−1.568	4.987	4.743	3.827
	4	1.098	9.262	−0.716	2.686	34.022

Table 2: F_{ACD} in units of its exact value (29): the dependence on the cutoffs s_0 and R. The dimensionless parameter is $r = 0.4$.

		$R = 3m_V^2$		$R = 5m_V^2$		$R = 8m_V^2$	
$s_0(m_V^2)$	N	P_n-fit	V_n-fit	P_n-fit	V_n-fit	P_n-fit	V_n-fit
0.2	0	0.691	0.510	0.479	0.298	0.338	0.183
	1	1.648	2.666	1.198	1.397	0.880	0.810
	2	1.346	4.058	1.564	3.273	1.346	1.984
	3	0.897	−0.728	1.375	4.278	1.527	3.478
	4	0.900	−7.144	1.018	1.504	1.404	4.351
0.3	0	0.609	0.529	0.428	0.304	0.305	0.186
	1	1.399	3.162	1.030	1.546	0.766	0.862
	2	1.256	5.750	1.350	4.068	1.155	2.282
	3	0.986	−0.726	1.269	6.146	1.327	4.401
	4	0.961	−14.043	1.062	2.975	1.282	6.176
0.4	0	0.554	0.549	0.392	0.311	0.282	0.188
	1	1.238	3.758	0.920	1.712	0.690	0.919
	2	1.188	8.256	1.211	5.053	1.031	2.621
	3	1.019	−0.826	1.192	8.887	1.197	5.555
	4	0.990	−22.961	1.069	6.380	1.196	8.775

dimension OPE coefficients may not improve the ACD-derived estimates, contrary to the claims in some early applications of the method.

The weighted routines should also be applied to more realistic model spectra, but our conclusions are expected to hold, at least qualitatively.

For a more reliable application of the ACD, we need a different approach to fitting routines. The most opportunistic approaches appear to be those with N fixed to unity, e. g. [14], but such estimates are intrinsically crude as they take into account only the first two terms of the OPE.

References

[1] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *Nucl. Phys.* **B147** (1979) 385

[2] N. F. Nasrallah, N. A. Papadopoulos and K. Schilcher, *Phys. Lett.* **B113** (1982) 61; **B126** (1983) 379; *Z. Phys.* **C16** (1983) 323.

[3] Siniša R. Ignjatović, *Analytic continuation by duality method in elementary particle physics*, PhD thesis, University of Cincinnati, 1997, and references therein.

[4] S. R. Ignjatović, in *Proc. IV Summer School in Modern Mathematical Physics*, SFIN (Belgrade) **20A**, 239 (2007).

[5] N. F. Nasrallah, *Phys. Rev.* **D63** (2001) paper 054028.

[6] S. R. Ignjatović, *Fortschr. Phys.* **56** (2008), 452.

[7] S. R. Ignjatović, T. Takeuchi, and L. C. R. Wijewardhana, *Phys. Rev.* **D61** (2000) paper 056006.

[8] A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics*, Birkhäuser, Boston, 1988.

[9] H. Bateman and A. Erdélyi, *Higher Transcedental Functions* v. 2, Mc-Graw Hill, New York, 1953.

[10] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.

[11] R. N. Desmarais and S. R. Bland, NASA RP-1343, 1995.

[12] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, 1964.

[13] M. Kramer et al., *Phys. Rev.* **D34** (1986) 2127.

[14] C. A. Dominguez et al., *J. High Energy Phys.* (2008) 0802, 072; 0805, 020.