Quantum enveloping algebras, von Neumann regularity and the Pierce decomposition

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Abstract

This work introduces quantum bialgebras, which differ from the standard $U_q(\mathfrak{sl}_2)$ by presence of von Neumann regular Cartan-like generators and the associated idempotents. Both invertible and von Neumann regular antipodes on such bialgebras are presented explicitly; the latter case leads to a von Neumann-Hopf algebra structure. Also, explicit forms of some particular R-matrices (also invertible and von Neumann regular) are presented, and the latter respects the Pierce decomposition.

An important feature of supersymmetric algebraic structures is that their underlying algebras normally contain some zero divisors, in particular idempotents [1]. On the other hand, the process of quantization of commutative spaces and superspaces involve the techniques related to Hopf algebras [2]. This was our motive to render idempotents to some quantum algebras, to study their properties and the associated Pierce decompositions [3]. In particular, we introduce new quantum algebras which admit an embedding of $U_q(\mathfrak{sl}_2)$.

Let $q \in \mathbb{C}$ and $q \neq \pm 1, 0$. The quantum universal enveloping algebra $U_q(\mathfrak{sl}_2)$ is a unital associative algebra (to be denoted by $U_q^{(alg)}(\mathfrak{sl}_2)$) determined by

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its (Chevalley) generators k, k^{-1}, e, f, and the relations

$$k^{-1}k = \mathbf{1}, \qquad kk^{-1} = \mathbf{1}, \qquad ke = q^2 ek, \qquad kf = q^{-2} fk,$$
$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}. \tag{1}$$

It admits a Hopf algebra structure which makes it a Hopf algebra $U_q^{(Hopf)}(\mathfrak{sl}_2)$ as follows:

$$\Delta_0(k) = k \otimes k \tag{2}$$
$$\Delta_0(e) = \mathbf{1} \otimes e + e \otimes k, \tag{3}$$
$$\Delta_0(f) = f \otimes \mathbf{1} + k^{-1} \otimes f, \tag{4}$$
$$\mathsf{S}_0(k) = k^{-1}, \tag{5}$$
$$\mathsf{S}_0(e) = -ek^{-1}, \qquad \mathsf{S}_0(f) = -kf, \tag{6}$$
$$\varepsilon_0(k) = 1, \qquad \varepsilon_0(e) = \varepsilon_0(f) = 0. \tag{7}$$

$U_q^{(alg)}(\mathfrak{sl}_2)$ is a domain: it has no zero divisors, hence no idempotents.

A basis of the vector space $U_q^{(alg)}(\mathfrak{sl}_2)$ is given by the monomials $k^s e^m f^n$, where $m, n \geq 0$.

A bialgebra $(H, \mu, \eta, \Delta, \varepsilon)$ over a field $\Bbbk$ is called a *von Neumann-Hopf algebra* if there exists $T \in \mathrm{Hom}_{\Bbbk}(H, H)$ such that $\mathrm{id} \star T \star \mathrm{id} = \mathrm{id}$ and $T \star \mathrm{id} \star T = T$, with $\star$ being the well known convolution product. Specifically, given two linear endomorphisms f and g on H, their convolution $f \star g$ is defined as follows:

$$(f \star g) \equiv \mu(f \otimes g)\Delta.$$

Note that, even if f and g here are endomorphisms of the algebra H, the resulting linear map $f \star g$ can fail to be an endomorphism of algebras. The T as above is called a *weak antipode*.

We start with a brief exposition of the previous results of Duplij and Li [5][1]. Although their considerations generalize the quantum universal enveloping algebra associated to an arbitrary semisimple Lie algebra $\mathfrak{g}$, we stick here to the case $\mathfrak{g} = \mathfrak{sl}_2$, which is consistent to our subsequent considerations.

Definition 1. *The algebra $\mathfrak{w}_q(\mathfrak{sl}_2)$ is given by its generators E, F, K, $\overline{K}$, J, and the relations*

$$J = K\overline{K}, \quad K\overline{K} = \overline{K}K, \quad JK = K, \quad J\overline{K} = \overline{K}, \quad KE = q^2 EK,$$
$$E\overline{K} = q^2 \overline{K}E, \quad KF = q^{-2} FK, \quad F\overline{K} = q^{-2}\overline{K}F, \quad EF - FE = \frac{K - \overline{K}}{q - q^{-1}}.$$

[1] The algebras as above were called weak Hopf algebras in [5]. It became clear later on that this term is not really good in the context and is reserved after a different subject.

It turns out that J is a central idempotent in $\mathfrak{w}_q(\mathfrak{sl}_2)$. This implies, in particular, that $\mathfrak{w}_q(\mathfrak{sl}_2)$ splits as a direct sum of two subalgebras $w_q = \mathfrak{w}_q(\mathfrak{sl}_2)J$ and $\overline{w}_q = \mathfrak{w}_q(\mathfrak{sl}_2)(1-J)$. However, this is not a direct sum of bialgebras, with the structure of coalgebra being described below.

$$\begin{aligned}
&\Delta(K) = K \otimes K, \quad \Delta(\overline{K}) = \overline{K} \otimes \overline{K}, \quad \Delta(J) = J \otimes J, \\
&\Delta(E) = 1 \otimes E + E \otimes K, \qquad \Delta(F) = F \otimes 1 + \overline{K} \otimes F, \\
&\varepsilon(E) = \varepsilon(F) = 0, \qquad \varepsilon(K) = \varepsilon(\overline{K}) = 1, \\
&T(1) = 1, \qquad T(E) = -E\overline{K}, \qquad T(F) = -KF, \\
&T(K) = \overline{K}, \qquad T(\overline{K}) = K, \qquad T(J) = J.
\end{aligned}$$

Theorem 2. $\mathfrak{w}_q(\mathfrak{sl}_2)$ *is a non-commutative and non-cocommutative von Neumann-Hopf algebra with the weak antipode T, but not a Hopf algebra. Moreover, $w_q \cong U_q^{(Hopf)}(\mathfrak{sl}_2)$ as Hopf algebras.*

We intend to produce some more suitable bialgebras, whose specific relation to $\mathfrak{w}_q(\mathfrak{sl}_2)$ will become clear later.

Let us denote the Cartan subalgebra of $U_q(\mathfrak{sl}_2)$ by $\mathcal{H}_0\left(\mathbf{1}, k, k^{-1}\right)$. We are going to generalize it towards the von Neumann regularity property.

Let us consider the generators K, $\overline{K}$ satisfying the Neumann regularity

$$K\overline{K}K = K, \qquad \overline{K}K\overline{K} = \overline{K}, \tag{8}$$

and the commutativity assumption

$$K\overline{K} = \overline{K}K. \tag{9}$$

Then one has an idempotent

$$P \overset{def}{=} K\overline{K} = \overline{K}K, \qquad P^2 = P$$

such that

$$PK = KP = K, \qquad P\overline{K} = \overline{K}P = \overline{K}.$$

The commutative algebra generated by K, $\overline{K}$ is not unital; we denote it by $\mathcal{H}(K, \overline{K})$.

$\mathcal{H}(K, \overline{K})$ was considered as a Cartan-like part of the analog of quantum enveloping algebra with von Neumann regular antipode $U_q^v = \mathfrak{vsl}_q(2)$ introduced by Duplij and Li [5, 8]. The associated *unital* algebra derived by an exterior attachment of unit $\mathcal{H}(\mathbf{1}, K, \overline{K}) \overset{def}{=} \mathcal{H}(K, \overline{K}) \oplus \mathbb{C}\mathbf{1}$ also appeared as a part of $U_q^w = \mathfrak{wsl}_q(2)$.

Observe that $\mathcal{H}(\mathbf{1}, K, \overline{K})$ contains one more idempotent $\mathbf{1} - P$, and our main idea is to introduce another copy of the same algebra (we denote it by $\mathcal{H}(L, \overline{L})$) subject to the relations

$$L\overline{L}L - L = 0, \qquad \overline{L}L\overline{L} - \overline{L} = 0, \qquad L\overline{L} = \overline{L}L. \tag{10}$$

The idempotent $Q \stackrel{def}{=} L\overline{L} = \overline{L}L$ satisfies

$$QL = LQ = L, \qquad Q\overline{L} = \overline{L}Q = \overline{L}.$$

With no additional relations between $K, \overline{K}$ and $L, \overline{L}$, the *nonunital* algebras $\mathcal{H}(K, \overline{K})$ and $\mathcal{H}(L, \overline{L})$ can form a *free product* only. We merge them to form a *unital* algebra $\mathcal{H}(\mathbf{1}, K, \overline{K}, L, \overline{L})$ and add one more relation, the decomposition of unity

$$P + Q = \mathbf{1}. \tag{11}$$

This produces a Pierce decomposition of the algebra $\mathcal{H}(\mathbf{1}, K, \overline{K}, L, \overline{L})$, which is nothing more than a direct product since

$$QP = (1 - P)P = PQ = P(1 - P) = 0.$$

It follows that

$$KL = \overline{L}K = LK = K\overline{L} = \overline{K}L = L\overline{K} = 0.$$

The new, as compared to Duplij and Li [5, 8], noninvertible generators L, $\overline{L}$ are justified by

Lemma 3. *$aK + bL$ is invertible, and its inverse is $a^{-1}\overline{K} + b^{-1}\overline{L}$, where $a, b \in \mathbb{R}\diagdown 0$.*

Let us consider a two-parameter family of morphisms for the Cartan subalgebra

$$\mathbf{\Phi}_{\mathcal{H}}^{(a,b)} : \mathcal{H}_0(\mathbf{1}, k, k^{-1}) \to \mathcal{H}(\mathbf{1}, K, \overline{K}, L, \overline{L})$$

given by

$$k \mapsto aK + bL, \qquad k^{-1} \mapsto a^{-1}\overline{K} + b^{-1}\overline{L}.$$

Proposition 4. *The map $\mathbf{\Phi}_{\mathcal{H}}^{(a,b)}$ is an embedding.*

Now we add two more generators E and F, along with the additional relations

$$(aK + bL)E = q^2 E(aK + bL), \tag{12}$$

$$(a^{-1}\overline{K} + b^{-1}\overline{L})E = q^{-2} E(a^{-1}\overline{K} + b^{-1}\overline{L}), \tag{13}$$

$$(aK + bL)F = q^{-2} F(aK + bL), \tag{14}$$

$$(a^{-1}\overline{K} + b^{-1}\overline{L})F = q^2 F(a^{-1}\overline{K} + b^{-1}\overline{L}), \tag{15}$$

$$EF - FE = \frac{(aK + bL) - (a^{-1}\overline{K} + b^{-1}\overline{L})}{q - q^{-1}} \tag{16}$$

which, together with (8) – (9) and (10) determine $U^{(alg)22}_{aK+bL}$. The indices 22 here stand for the number of generators between the Cartan-like generators $(K,\ L)$ and $E,\ F$. This algebra corresponds to $U^w_q = \mathfrak{wsl}_q(2)$ introduced by Duplij and Li [5, 8]. To be more precise, there exists an algebra homomorphism $\mathfrak{wsl}_q(2) \to U^{(alg)22}_{aK+bL}$, which in the notation of [5] is given by

$$K_w \mapsto aK{+}bL, \qquad \overline{K}_w \mapsto a^{-1}\overline{K}{+}b^{-1}\overline{L}, \qquad E_w \mapsto E, \qquad F_w \mapsto F. \tag{17}$$

As one can see from Lemma 3, together with (12) – (16), the image of this homomorphism is a copy of $U_q(\mathfrak{sl}_2)$, cf. [5, Proposition 1].

In a similar way, we produce an analog of $U^v_q = \mathfrak{vsl}_q(2)$ from [5, 8] by introducing the algebra with the same generators as $U^{(alg)22}_{aK+bL}$, but subject to the relations as follows:

$$(aK + bL)E(a^{-1}\overline{K} + b^{-1}\overline{L}) = q^2 E, \tag{18}$$

$$(a^{-1}\overline{K} + b^{-1}\overline{L})E(aK + bL) = q^{-2} E, \tag{19}$$

$$(aK + bL)F(a^{-1}\overline{K} + b^{-1}\overline{L}) = q^{-2} F, \tag{20}$$

$$(a^{-1}\overline{K} + b^{-1}\overline{L})F(aK + bL) = q^2 F, \tag{21}$$

$$EF - FE = \frac{(aK + bL) - (a^{-1}\overline{K} + b^{-1}\overline{L})}{q - q^{-1}}, \tag{22}$$

which we denote by $U^{(alg)31}_{aK+bL}$. This algebra corresponds to the algebra $U^v_q = \mathfrak{vsl}_q(2)$ [5] in the sense that there exists an algebra homomorphism $\mathfrak{vsl}_q(2) \to U^{(alg)31}_{aK+bL}$. Again, this homomorphism, in the notation of [5], is given on the generators by (17), with the indices w being replaced by v. Another application of Lemma 3 allows one to observe that the image of this homomorphism is a copy of $U_q(\mathfrak{sl}_2)$, cf. [5, Proposition 1].

Let us introduce an extension $\mathbf{\Phi}^{(a,b)}$ of $\mathbf{\Phi}^{(a,b)}_{\mathcal{H}}$ to the entire algebras $U^{(alg)22}_{aK+bL}$ and $U^{(alg)31}_{aK+bL}$:

$$k \mapsto aK + bL, \qquad k^{-1} \mapsto a^{-1}\overline{K} + b^{-1}\overline{L}, \qquad e \mapsto E, \qquad f \mapsto F.$$

Proposition 5. *The algebras $U^{(alg)22}_{aK+bL}$ and $U^{(alg)31}_{aK+bL}$ are isomorphic to $U^{(alg)22}_{K+L} \overset{def}{=} U^{(alg)22}_{aK+bL}\Big|_{a=1,b=1}$ and $U^{(alg)31}_{K+L} \overset{def}{=} U^{(alg)31}_{aK+bL}\Big|_{a=1,b=1}$, respectively.*

In view of this proposition, we remove the parameters a and b from the subsequent considerations.

The idempotents P and Q are not central in $U^{(alg)22}_{K+L}$ and $U^{(alg)31}_{K+L}$. We can "split" the relations (12) – (16) and (18) – (22) in such a way that P and

Q become central

$$PE = EP, \qquad QE = EQ, \qquad PF = FP, \qquad QF = FQ,$$

or satisfy the "twisting" conditions

$$PE = EQ, \qquad QE = EP, \qquad PF = FQ, \qquad QF = FP.$$

Thus, we obtain the "splitted" 22-algebras as follows. The defining relations for $U^{(alg)22}_{K,L,norm}$ are

$$\begin{array}{llll} K\overline{K}K = K, & \overline{K}K\overline{K} = \overline{K}, & K\overline{K} = \overline{K}K, & \\ L\overline{L}L = L, & \overline{L}L\overline{L} = \overline{L}, & L\overline{L} = \overline{L}L, & \\ KE = q^2EK, & \overline{K}E = q^{-2}E\overline{K}, & LE = q^2EL, & \overline{L}E = q^{-2}E\overline{L}, \\ KF = q^{-2}FK, & \overline{K}F = q^2F\overline{K}, & LF = q^{-2}FL, & \overline{L}F = q^2F\overline{L}, \\ EF - FE = \dfrac{(K+L) - (\overline{K}+\overline{L})}{q-q^{-1}}. & & & \end{array} \tag{23}$$

Another "splitted" 22-algebra $U^{(alg)22}_{K,L,twist}$ is given by

$$\begin{array}{llll} K\overline{K}K = K, & \overline{K}K\overline{K} = \overline{K}, & K\overline{K} = \overline{K}K, & \\ L\overline{L}L = L, & \overline{L}L\overline{L} = \overline{L}, & L\overline{L} = \overline{L}L, & \\ KE = q^2EL, & \overline{K}E = q^{-2}E\overline{L}, & LE = q^2EK, & \overline{L}E = q^{-2}E\overline{K}, \\ KF = q^{-2}FL, & \overline{K}F = q^2F\overline{L}, & LF = q^{-2}FK, & \overline{L}F = q^2F\overline{K}, \\ EF - FE = \dfrac{(K+L) - (\overline{K}+\overline{L})}{q-q^{-1}}. & & & \end{array} \tag{24}$$

Similarly, we describe the "splitted" 31-algebras. The defining relations for $U^{(alg)31}_{K,L,norm}$ are

$$\begin{array}{llll} K\overline{K}K = K, & \overline{K}K\overline{K} = \overline{K}, & K\overline{K} = \overline{K}K = P, & \\ L\overline{L}L = L, & \overline{L}L\overline{L} = \overline{L}, & L\overline{L} = \overline{L}L = Q, & \\ KE\overline{K} = q^2EP, & \overline{K}EK = q^{-2}EP, & LE\overline{L} = q^2EQ, & \overline{L}EL = q^{-2}EQ, \\ KF\overline{K} = q^{-2}FP, & \overline{K}FK = q^2FP, & LF\overline{L} = q^{-2}FQ, & \overline{L}FL = q^2FQ, \\ P(EF - FE) = \dfrac{K-\overline{K}}{q-q^{-1}}, & & Q(EF - FE) = \dfrac{L-\overline{L}}{q-q^{-1}}, & \end{array}$$

and for $U^{(alg)31}_{K,L,twist}$

$$\begin{array}{llll} K\overline{K}K = K, & \overline{K}K\overline{K} = \overline{K}, & K\overline{K} = \overline{K}K = P, & \\ L\overline{L}L = L, & \overline{L}L\overline{L} = \overline{L}, & L\overline{L} = \overline{L}L = Q, & \\ KE\overline{L} = q^2EQ, & \overline{K}EL = q^{-2}EQ, & LE\overline{K} = q^2EP, & \overline{L}EK = q^{-2}EP, \\ KF\overline{L} = q^{-2}FQ, & \overline{K}FL = q^2FQ, & LF\overline{K} = q^{-2}FP, & \overline{L}FK = q^2FP, \end{array}$$

$$P(EF - FE) = \frac{K - \overline{K}}{q - q^{-1}}, \qquad Q(EF - FE) = \frac{L - \overline{L}}{q - q^{-1}}.$$

Proposition 6. *One has the isomorphisms:* $U^{(alg)22}_{K,L,norm} \cong U^{(alg)31}_{K,L,norm}$, *and* $U^{(alg)22}_{K,L,twist} \cong U^{(alg)31}_{K,L,twist}$.

Therefore, in what follows we consider the algebras $U^{(alg)22}_{K,L,norm}$, $U^{(alg)22}_{K,L,twist}$ (with the 22 superscript being discarded) only.

Now we extend $\mathbf{\Phi}_{\mathcal{H}}$ to a morphism taking values in the "splitted" algebras $U^{(alg)}_{K,L,norm}$ and $U^{(alg)}_{K,L,twist}$ as follows

$$\mathbf{\Phi} : \begin{cases} k \mapsto K + L, & k^{-1} \mapsto \overline{K} + \overline{L}, \\ e \mapsto E, & f \mapsto F. \end{cases} \tag{25}$$

Proposition 7. *The map $\mathbf{\Phi}$ defined on the generators as above, admits an extension to a well defined morphism of algebras from $U_q(\mathfrak{sl}_2)$ to either $U^{(alg)}_{K,L,norm}$ or $U^{(alg)}_{K,L,twist}$, which is an embedding.*

Corollary 8. *Both algebras $U^{(alg)}_{K,L,norm}$ and $U^{(alg)}_{K,L,twist}$ contain $U_q(\mathfrak{sl}_2)$ as a subalgebra.*

The Pierce decomposition of $U^{(alg)}_{K,L,norm}$ is

$$U^{(alg)}_{K,L,norm} = PU^{(alg)}_{K,L,norm}P + QU^{(alg)}_{K,L,norm}Q, \tag{26}$$

which reduces to a direct sum of the two ideals. This leads to

Proposition 9. *$U^{(alg)}_{K,L,norm}$ is a direct sum of subalgebras with each summand being isomorphic to $U_q(\mathfrak{sl}_2)$.*

In the "twisted" case the Pierce decomposition

$$\begin{aligned} U^{(alg)}_{K,L,twist} = PU^{(alg)}_{K,L,twist}P + PU^{(alg)}_{K,L,twist}Q \\ + QU^{(alg)}_{K,L,twist}P + QU^{(alg)}_{K,L,twist}Q, \end{aligned} \tag{27}$$

is nontrivial as all terms are nonzero, i.e. it is not a direct sum of ideals.

Proposition 10. *The Poincaré-Birkhoff-Witt basis of $U_{K,L,norm}^{(alg)}$ is given by the monomials*

$$\left[\left\{PK^iE^jF^k\right\}_{i,j,k\geq 0}\cup\left\{\overline{K}^iE^jF^k\right\}_{i>0,j,k\geq 0}\right]$$
$$\cup\left[\left\{QL^iE^jF^k\right\}_{i,j,k\geq 0}\cup\left\{\overline{L}^iE^jF^k\right\}_{i>0,j,k\geq 0}\right].$$

In the case of $U_{K,L,twist}^{(alg)}$ we have a direct sum decomposition into 4 vector subspaces. We present below a PBW basis which respects this decomposition.

Proposition 11. *The Poincaré-Birkhoff-Witt basis of $U_{K,L,twist}^{(alg)}$ is given by the monomials*

$$\left[\left\{PK^iE^jF^k\right\}_{\substack{i,j,k\geq 0\\ j+k\ even}}\cup\left\{\overline{K}^iE^jF^k\right\}_{\substack{i>0,j,k\geq 0\\ j+k\ even}}\right]$$
$$\cup\left[\left\{PK^iE^jF^k\right\}_{\substack{i,j,k\geq 0\\ j+k\ odd}}\cup\left\{\overline{K}^iE^jF^k\right\}_{\substack{i>0,j,k\geq 0\\ j+k\ odd}}\right]$$
$$\cup\left[\left\{QL^iE^jF^k\right\}_{\substack{i,j,k\geq 0\\ j+k\ odd}}\cup\left\{\overline{L}^iE^jF^k\right\}_{\substack{i>0,j,k\geq 0\\ j+k\ odd}}\right]$$
$$\cup\left[\left\{QL^iE^jF^k\right\}_{\substack{i,j,k\geq 0\\ j+k\ even}}\cup\left\{\overline{L}^iE^jF^k\right\}_{\substack{i>0,j,k\geq 0\\ j+k\ even}}\right].$$

Let us consider the classical limit $q\to 1$ for $U_{K,L,norm}^{(alg)}$ and $U_{K,L,twist}^{(alg)}$ algebras.

Proposition 12. *The classical limit of $U_{K,L,norm}^{(alg)}$ is just a direct sum of two copies of the classical limits for $U_q(\mathfrak{sl}_2)$ in the sense of Kassel [7].*

We apply the embedding $\mathbf{\Phi}$ to the standard relations (2), (4), (7) in $U_q^{(alg)}(\mathfrak{sl}_2)$ to transfer a coproduct and a counit onto the image of $\mathbf{\Phi}$ as follows

$$\Delta(K+L)=(K+L)\otimes(K+L), \tag{28}$$
$$\Delta(\overline{K}+\overline{L})=(\overline{K}+\overline{L})\otimes(\overline{K}+\overline{L}), \tag{29}$$
$$\Delta(E)=\mathbf{1}\otimes E+E\otimes(K+L), \tag{30}$$
$$\Delta(F)=F\otimes\mathbf{1}+(\overline{K}+\overline{L})\otimes F, \tag{31}$$
$$\varepsilon(E)=\varepsilon(F)=0, \tag{32}$$
$$\varepsilon(K+L)=1, \tag{33}$$
$$\varepsilon(\overline{K}+\overline{L})=1. \tag{34}$$

To construct a counit ε on the "splitted" algebras $U^{(alg)}_{K,L,norm}$ and $U^{(alg)}_{K,L,twist}$, observe that since P and Q are idempotents, one has

$$\varepsilon(P)(\varepsilon(P)-1)=0, \qquad \varepsilon(Q)(\varepsilon(Q)-1)=0,$$

which implies that either

$$1)\ \varepsilon(P)=1, \quad \varepsilon(Q)=0 \qquad \text{or} \qquad 2)\ \varepsilon(P)=0, \quad \varepsilon(Q)=1.$$

We assume the first choice for both $U^{(alg)}_{K,L,norm}$ and $U^{(alg)}_{K,L,twist}$. Then it follows from $L=QL$ that $\varepsilon(L)=\varepsilon(QL)=0$. Also it follows from (33) that $\varepsilon(K)=1$.

In a similar way, to produce a comultiplication on $U^{(alg)}_{K,L,norm}$ and $U^{(alg)}_{K,L,twist}$ determined by (23) and (24), respectively, use (28) – (34) to define a coproduct Δ first on $\mathbf{\Phi}(U^{(alg)}_q(\mathfrak{sl}_2))$ (via transferring from $U^{(alg)}_q(\mathfrak{sl}_2)$) and then extend it to the entire algebras $U^{(alg)}_{K,L,norm}$ and $U^{(alg)}_{K,L,twist}$ as follows. In the case of $U^{(coalg)}_{K,L,norm}$ one has:

$$\begin{aligned}
&\Delta(K)=K\otimes K, && \Delta(\overline{K})=\overline{K}\otimes\overline{K},\\
&\Delta(L)=L\otimes L+L\otimes K+K\otimes L, && \Delta(\overline{L})=\overline{L}\otimes\overline{L}+\overline{L}\otimes\overline{K}+\overline{K}\otimes\overline{L},\\
&\Delta(E)=\mathbf{1}\otimes E+E\otimes(K+L), && \Delta(F)=F\otimes\mathbf{1}+(\overline{K}+\overline{L})\otimes F,\\
&\varepsilon(E)=\varepsilon(F)=\varepsilon(L)=\varepsilon(\overline{L})=0, && \varepsilon(K)=\varepsilon(\overline{K})=1,
\end{aligned}$$

and in the case of $U^{(coalg)}_{K,L,twist}$

$$\begin{aligned}
&\Delta(K)=K\otimes K+L\otimes L, && \Delta(\overline{K})=\overline{K}\otimes\overline{K}+\overline{L}\otimes\overline{L},\\
&\Delta(L)=L\otimes K+K\otimes L, && \Delta(\overline{L})=\overline{L}\otimes\overline{K}+\overline{K}\otimes\overline{L}\\
&\Delta(E)=\mathbf{1}\otimes E+E\otimes(K+L), && \Delta(F)=F\otimes\mathbf{1}+(\overline{K}+\overline{L})\otimes F,\\
&\varepsilon(E)=\varepsilon(F)=\varepsilon(L)=\varepsilon(\overline{L})=0, && \varepsilon(K)=\varepsilon(\overline{K})=1.
\end{aligned}$$

Proposition 13. *The bialgebra $U^{(bialg)}_{K,L,norm}$ has no conventional antipode S satisfying the standard Hopf algebra axiom*

$$\mathsf{S}\star\mathsf{id}=\mathsf{id}\star\mathsf{S}=\eta\circ\varepsilon.$$

Introduce an antimorphism T by

$$\mathsf{T}(K)=\overline{K}, \qquad \mathsf{T}(\overline{K})=K, \tag{35}$$

$$\mathsf{T}(L)=\overline{L}, \qquad \mathsf{T}(\overline{L})=L, \tag{36}$$

$$\mathsf{T}(E)=-E(\overline{K}+\overline{L}), \qquad \mathsf{T}(F)=-(K+L)F. \tag{37}$$

For $U^{(bialg)}_{K,L,norm}$ we observe that

$$\begin{gathered}(\mathsf{T}\star\mathsf{id})(K)=(\mathsf{id}\star\mathsf{T})(K)=(\mathsf{T}\star\mathsf{id})(\overline{K})=(\mathsf{id}\star\mathsf{T})(\overline{K})=P,\\ (\mathsf{T}\star\mathsf{id})(L)=(\mathsf{id}\star\mathsf{T})(L)=(\mathsf{T}\star\mathsf{id})(\overline{L})=(\mathsf{id}\star\mathsf{T})(\overline{L})=Q,\\ (\mathsf{T}\star\mathsf{id})(E)=(\mathsf{id}\star\mathsf{T})(E)=(\mathsf{T}\star\mathsf{id})(F)=(\mathsf{id}\star\mathsf{T})(F)=0.\end{gathered}$$

Proposition 14. *The antimorphism* T *is von Neumann regular on any element of the bialgebra* $U^{(bialg)}_{K,L,norm}$

$$\mathsf{id}\star\mathsf{T}\star\mathsf{id}=\mathsf{id},\qquad \mathsf{T}\star\mathsf{id}\star\mathsf{T}=\mathsf{T}.$$

Definition 15. *We call the antimorphism* T *a von Neumann regular antipode of* $U^{(bialg)}_{K,L,norm}$.

Definition 16. *We call a bialgebra with a von Neumann regular antipode a von Neumann-Hopf algebra.*

Remark 17. *The standard Drinfeld-Jimbo algebra* $U_q(\mathfrak{sl}_2)$ *(which is a domain) admits no embedding of* $U^{(bialg)}_{K,L,norm}$*, because the latter contains zero divisors (e.g.* (11)*).*

Let us consider a possibility to produce a Hopf algebra structure on $U^{(bialg)}_{K,L,twist}$. Introduce an antimorphism S of $U^{(bialg)}_{K,L,twist}$ by the same formulas as (35) – (37):

$$\begin{gathered}\mathsf{S}(K)=\overline{K},\quad \mathsf{S}(\overline{K})=K,\quad \mathsf{S}(L)=\overline{L},\quad \mathsf{S}(\overline{L})=L,\\ \mathsf{S}(E)=-E(\overline{K}+\overline{L}),\qquad \mathsf{S}(F)=-(K+L)F.\end{gathered}$$

We have for $U^{(bialg)}_{K,L,twist}$

$$\begin{gathered}(\mathsf{id}\star\mathsf{S})(K)=(\mathsf{S}\star\mathsf{id})(K)=(\mathsf{S}\star\mathsf{id})(\overline{K})=(\mathsf{id}\star\mathsf{S})(\overline{K})=\mathbf{1},\\ (\mathsf{id}\star\mathsf{S})(L)=(\mathsf{S}\star\mathsf{id})(L)=(\mathsf{S}\star\mathsf{id})(\overline{L})=(\mathsf{id}\star\mathsf{S})(\overline{L})=0,\\ (\mathsf{id}\star\mathsf{S})(E)=(\mathsf{S}\star\mathsf{id})(E)=(\mathsf{S}\star\mathsf{id})(F)=(\mathsf{id}\star\mathsf{S})(F)=0.\end{gathered}$$

The proof of the following statement is basically due to [6, p.35].

Proposition 18. *The relations* $(\mathsf{id} \star \mathsf{S})(X) = (\mathsf{S} \star \mathsf{id})(X) = \varepsilon(X) \cdot \mathbf{1}$ *are valid for any* $X \in U^{(bialg)}_{K,L,twist}$.

Thus, we have the following

Theorem 19. *1)* $U^{(Hopf)}_{K,L} \stackrel{def}{=} (U^{(bialg)}_{K,L,twist}, \mathsf{S})$ *is a Hopf algebra;*

2) $U^{(vN-Hopf)}_{K,L} \stackrel{def}{=} (U^{(bialg)}_{K,L,norm}, \mathsf{T})$ *is a von Neumann-Hopf algebra.*

Let us consider a version of universal R-matrix for $U^{(vN-Hopf)}_{K,L}$ and $U^{(Hopf)}_{K,L}$. In order to avoid considerations related to formal series (the general context of R-matrices), we turn to quasi-cocommutative bialgebras [7]. Such bialgebras generate R-matrices of some simpler shape admitting (under some additional assumptions) an explicit formula to be described below.

Definition 20. *A bialgebra* $U^{(bialg)} = (\mathbb{C}, B, \mu, \eta, \Delta, \varepsilon)$ *is called quasi-cocommutative, if there exists an invertible element* $R \in U^{(bialg)} \otimes U^{(bialg)}$, *called a universal R-matrix, such that*

$$\Delta^{\mathrm{cop}}(b) = R\Delta(b)R^{-1}, \quad \forall b \in U^{(bialg)},$$

where Δ^{cop} *is the opposite comultiplication in* $U^{(bialg)}$.

The R-matrix of a braided bialgebra $U^{(bialg)}$ is subject to

$$(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}, \qquad (\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}, \tag{38}$$

where for $R = \sum_i s_i \otimes t_i$ one has $R_{12} = \sum_i s_i \otimes t_i \otimes \mathbf{1}$, etc. [4]. From now on we assume that $q^n = 1$, which is a distinct case to the above context.

Consider the two-sided ideal I_{sl_2} in $U^{(alg)}_q(\mathfrak{sl}_2)$ generated by $\{k^n - \mathbf{1}, e^n, f^n\}$, together with the associated quotient algebra $\widehat{U}^{(alg)}_q(\mathfrak{sl}_2) = U^{(alg)}_q(\mathfrak{sl}_2) \big/ I_{sl_2}$.

Theorem 21 ([7, p. 230]). *The universal R-matrix of* $\widehat{U}^{(alg)}_q(\mathfrak{sl}_2)$ *is*

$$\widehat{R} = \sum_{0 \le i,j,m \le n-1} A^{ij}_m(q) \cdot e^m k^i \otimes f^m k^j, \tag{39}$$

$$A^{ij}_m(q) = \frac{1}{n} \frac{(q - q^{-1})^m}{[m]!} q^{\frac{m(m-1)}{2} + 2m(i-j) - 2ij}, \tag{40}$$

where $[m]! = [1][2] \ldots [m]$, $[m] = (q^m - q^{-m}) \big/ (q - q^{-1})$.

Now we use (25) to obtain an analog of this theorem for $U^{(Hopf)}_{K,L}$. In a similar way, we consider the quotient algebra $\widehat{U}^{(Hopf)}_{K+L} = U^{(Hopf)}_{K,L} \Big/ I^{(Hopf)}_{K+L}$, where the two-sided ideal $I^{(Hopf)}_{K+L}$ is generated by $\{K^n + L^n - \mathbf{1}, E^n, F^n\}$.

Theorem 22. *The universal R-matrix of* $\widehat{U}_{K,L}^{(Hopf)}$ *is given by*

$$\widehat{R}_{K+L}^{(Hopf)} = \sum_{0 \le i,j,m \le n-1} A_m^{ij}(q) \cdot E^m (K^i + L^i) \otimes F^m (K^j + L^j).$$

Turn to writing down an explicit form for the universal R-matrix in the case of $U_{K,L}^{(vN-Hopf)}$. Again we consider the quotient algebra $\widehat{U}_{K+L}^{(vN-Hopf)} = U_{K,L}^{(vN-Hopf)} \Big/ I_{K+L}^{(vN-Hopf)}$, where the two-sided ideal $I_{K,L}^{(vN-Hopf)}$ is generated by $\{K^n + L^n - \mathbf{1}, E^n, F^n\}$.

Theorem 23. *The universal R-matrix of* $\widehat{U}_{K+L}^{(vN-Hopf)}$ *is given by*

$$\widehat{R}_{K+L}^{(vN-Hopf)} = \sum_{0 \le i,j,m \le n-1} A_m^{ij}(q) \cdot E^m (K^i + L^i) \otimes F^m (K^j + L^j). \tag{41}$$

Remark 24. *In view of* ***Theorem 21*** *the R-matrices we have introduced satisfy the Yang-Baxter equation by our construction.*

Note that $\widehat{R}_{K+L}^{(vN-Hopf)}$ is not compatible to the direct sum decomposition (26). Now we present another notion of R-matrix which respects (26), but differs from that described in **Definition 20** in the sense of being noninvertible.

Definition 25. *A bialgebra* $\widetilde{U}^{(bialg)} = (\mathbb{C}, B, \mu, \eta, \Delta, \varepsilon)$ *is called near-quasi-cocommutative, if there exists an element* $\widetilde{R} \in \widetilde{U}^{(bialg)} \otimes \widetilde{U}^{(bialg)}$*, called a universal near-R-matrix, such that*

$$\Delta^{\mathrm{cop}}(b)\widetilde{R} = \widetilde{R}\Delta(b), \qquad \forall b \in \widetilde{U}^{(bialg)},$$

where Δ^{cop} *is the opposite comultiplication in* $\widetilde{U}^{(bialg)}$ *and an element* $\widetilde{R}^{\dagger} \in \widetilde{U}^{(bialg)} \otimes \widetilde{U}^{(bialg)}$ *is such that*

$$\widetilde{R}\widetilde{R}^{\dagger}\widetilde{R} = \widetilde{R}, \qquad \widetilde{R}^{\dagger}\widetilde{R}\widetilde{R}^{\dagger} = \widetilde{R}^{\dagger},$$

and $\widetilde{R}^{\dagger}$ *can be named the Moore-Penrose inverse for a near-R-matrix [9, 10].*

Definition 26. *A near-quasi-cocommutative bialgebra* $\widetilde{U}^{(bialg)}$ *is braided, if its near-R-matrix satisfies* (38).

Consider the quotient algebra $\widehat{U}_{K,L}^{(vN-Hopf)} = U_{K,L}^{(vN-Hopf)} \Big/ I_{K,L}^{(vN-Hopf)}$, where the two-sided ideal $I_{K,L}^{(vN-Hopf)}$ is generated by $\{K^n - P, L^n - Q, E^n, F^n\}$.

Theorem 27. *The universal R-matrix of* $\widehat{U}_{K,L}^{(vN-Hopf)}$ *is given by the sum*

$$\widehat{R}_{K,L}^{(vN-Hopf)} = \widehat{R}_{PP}^{(vN-Hopf)} + \widehat{R}_{QQ}^{(vN-Hopf)}, \qquad \textit{where}$$

$$\widehat{R}_{PP}^{(vN-Hopf)} = \sum_{0 \le i,j,m \le n-1} A_m^{ij}(q) \cdot E^m K^i \otimes F^m K^j, \tag{42}$$

$$\widehat{R}_{QQ}^{(vN-Hopf)} = \sum_{0 \le i,j,m \le n-1} A_m^{ij}(q) \cdot E^m L^i \otimes F^m L^j. \tag{43}$$

Remark 28. *The universal near-R-matrix* $\widehat{R}_{K,L}^{(vN-Hopf)}$ *can be presented in the form*

$$\widehat{R}_{K,L}^{(vN-Hopf)} = (P \otimes P)\widehat{R}_{PP}^{(vN-Hopf)} + (Q \otimes Q)\widehat{R}_{QQ}^{(vN-Hopf)}. \tag{44}$$

Turn to presenting an example of representations for $U_{K,L,norm}^{(alg)}$ and $U_{K,L,twist}^{(alg)}$ associated to a given finite-dimensional representation of $U_q(\mathfrak{sl}_2)$. Let V be an n-dimensional vector space and $\pi : U_q(\mathfrak{sl}_2) \to \operatorname{End} V$ is one of the representations as in [7]. Consider a vector space $V \oplus V$ of double dimension. Form a representation $\Pi_{norm} : U_{K,L,norm}^{(alg)} \to \operatorname{End}(V \oplus V)$ of $U_{K,L,norm}^{(alg)}$ as follows

$$\Pi_{norm}(K) = \begin{pmatrix} \pi(k) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Pi_{norm}(\overline{K}) = \begin{pmatrix} \pi(k^{-1}) & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Pi_{norm}(L) = \begin{pmatrix} 0 & 0 \\ 0 & \pi(k) \end{pmatrix}, \qquad \Pi_{norm}(\overline{L}) = \begin{pmatrix} 0 & 0 \\ 0 & \pi(k^{-1}) \end{pmatrix},$$

$$\Pi_{norm}(E) = \begin{pmatrix} \pi(e) & 0 \\ 0 & \pi(e) \end{pmatrix}, \quad \Pi_{norm}(F) = \begin{pmatrix} \pi(f) & 0 \\ 0 & \pi(f) \end{pmatrix}.$$

An easy computation demonstrates that the above representation is well-defined and in particular is compatible to (1) (23). In a similar way define a representation $\Pi_{twist} : U_{K,L,twist}^{(alg)} \to \operatorname{End}(V \oplus V)$ of $U_{K,L,twist}^{(alg)}$ by

$$\Pi_{twist}(K) = \begin{pmatrix} \pi(k) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Pi_{twist}(\overline{K}) = \begin{pmatrix} \pi(k^{-1}) & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Pi_{twist}(L) = \begin{pmatrix} 0 & 0 \\ 0 & \pi(k) \end{pmatrix}, \qquad \Pi_{twist}(\overline{L}) = \begin{pmatrix} 0 & 0 \\ 0 & \pi(k^{-1}) \end{pmatrix},$$

$$\Pi_{twist}(E) = \begin{pmatrix} 0 & \pi(e) \\ \pi(e) & 0 \end{pmatrix}, \quad \Pi_{twist}(F) = \begin{pmatrix} 0 & \pi(f) \\ \pi(f) & 0 \end{pmatrix}.$$

Note that Π_{norm} is a direct sum of two representations, while the representation Π_{twist} is compatible to the Pierce decomposition of $U_{K,L,twist}^{(alg)}$ (27). It would be interesting to consider a representation theory of such algebras.

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