p-Adic, Adelic and Zeta Strings*

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ABSTRACT

We present a very brief review of p-adic, adelic and zeta strings. Details can be found in cited literature.

1. Introduction

p-Adic string theory emerged in 1987 [1] as a successful application of p-adic numbers in string theory. The most attractive p-adic strings have been those whose world sheet is p-adic and all other aspects are described by real or complex numbers. Four-point scattering amplitudes of open scalar ordinary and p-adic strings are connected at the tree level by their product, which is a constant. Similar product formula was found for closed scalar strings. These product formulas gave rise to introduce a notion of adelic strings, which are composed of ordinary and p-adic strings (see, e.g. [2, 3] for a review). Some other p-adic structures have been also investigated and p-adic mathematical physics was established (for a recent review we refer to [4]).

One of the main achievements in p-adic string theory is an effective nonlocal field description of p-adic scalar strings [5, 6]. The corresponding Lagrangian describes at the tree-level four-point scattering amplitudes and all higher ones. In the recent years the Lagrangian approach to p-adic string theory has been significantly advanced and many aspects of p-adic string dynamics have been considered, compared with dynamics of ordinary strings and applied to nonlocal cosmology (see, e.g. [7, 8, 9, 10, 11] and references therein).

Adelic approach has been applied to quantum mechanics [12], Feynman path integral [13], quantum cosmology [14], summation of divergent series [15], and dynamical systems [16].

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2. On p-Adic Strings

The crossing symmetric Veneziano amplitude for scattering of two ordinary open scalar strings is

$$
\begin{aligned}
A_\infty(a,b) &= g_\infty^2 \int_{\mathbb{R}} |x|_\infty^{a-1}\, |1-x|_\infty^{b-1}\, d_\infty x \\
&= g_\infty^2 \frac{\zeta(1-a)}{\zeta(a)}\, \frac{\zeta(1-b)}{\zeta(b)}\, \frac{\zeta(1-c)}{\zeta(c)}\,, \qquad (1)
\end{aligned}
$$

where $a = -\alpha(s) = -\frac{s}{2} - 1$, $b = -\alpha(t)$, $c = -\alpha(u)$ with the condition $a + b + c = 1$, i.e. $s + t + u = -8$. In (1), $|\cdot|_\infty$ denotes the ordinary absolute value, $\mathbb{R}$ is the field of real numbers, kinematic variables $a, b, c \in \mathbb{C}$, and ζ is the Riemann zeta function. The corresponding Veneziano amplitude for scattering of two p-adic strings was introduced as p-adic analog of the integral in (1), i.e.

$$
A_p(a,b) = g_p^2 \int_{\mathbb{Q}_p} |x|_p^{a-1}\, |1-x|_p^{b-1}\, d_p x\,, \qquad (2)
$$

where $\mathbb{Q}_p$ is the field of p-adic numbers, $|\cdot|_p$ is p-adic absolute value and $d_p x$ is the additive Haar measure on $\mathbb{Q}_p$ (see, e.g. [2, 3] and [17] for basic properties of p-adic numbers, adeles and their functions). Note that variable x in the integrands is related to the string world-sheet. Thus this kind of p-adic strings differ from ordinary one by p-adic treatment only of the world-sheet. Namely, in (2), kinematic variables a, b, c maintain their complex values with the same condition $a + b + c = 1$. Integrals in (1) and (2) are examples of Gel'fand-Graev-Tate beta functions on $\mathbb{R}$ and $\mathbb{Q}_p$, respectively. Integration in (2) gives

$$
A_p(a,b) = g_p^2\, \frac{1-p^{a-1}}{1-p^{-a}}\, \frac{1-p^{b-1}}{1-p^{-b}}\, \frac{1-p^{c-1}}{1-p^{-c}}\,, \qquad (3)
$$

where p is any prime number.

Scattering amplitude (3) can be obtained from the following Lagrangian of the effective scalar field φ [5, 6]:

$$
\mathcal{L}_p = \frac{m^D}{g_p^2}\, \frac{p^2}{p-1} \Big[-\frac{1}{2}\, \varphi\, p^{-\frac{\Box}{2m^2}}\, \varphi + \frac{1}{p+1}\, \varphi^{p+1} \Big]\,, \qquad (4)
$$

where $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alembertian.

Lagrangian (4) is nonlocal with an infinite number of spacetime derivatives. The equation of motion for (4) is

$$
p^{-\frac{\Box}{2m^2}}\, \varphi = \varphi^p\,, \qquad (5)
$$

and its properties have been studied by many authors (see, [9] and references therein).

3. On Adelic Strings

Recall the definition of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}\,, \quad s = \sigma + i\tau\,, \quad \sigma > 1\,, \tag{6}$$

which has analytic continuation to the entire complex s plane, excluding the point $s = 1$, where it has a simple pole with residue 1.

According to (6) one can take product of p-adic string amplitudes (3)

$$\prod_p A_p(a,b) = \frac{\zeta(a)}{\zeta(1-a)} \frac{\zeta(b)}{\zeta(1-b)} \frac{\zeta(c)}{\zeta(1-c)} \prod_p g_p^2\,, \tag{7}$$

that gives a nice simple formula

$$A_\infty(a,b) \prod_p A_p(a,b) = g_\infty^2 \prod_p g_p^2\,. \tag{8}$$

Note that product of all p-adic amplitudes in (7) is divergent, but can be made convergent after an appropriate regularization. To have product of amplitudes (8) finite it must be finite product of coupling constants, i.e. $g_\infty^2 \prod_p g_p^2 = const.$ There are three interesting possibilities for g_p^2: (i) $g_p^2 = 1$, (ii) $g_p^2 = \frac{p^2}{p^2-1}$, what gives $\prod_p g_p^2 = \zeta(2)$, (iii) $g_p^2 = |\frac{m}{n}|_p$, where m and n are any two nonzero integers, and it gives $g_\infty^2 \prod_p g_p^2 = |\frac{m}{n}|_\infty \prod_p |\frac{m}{n}|_p = 1$.

From (8) it follows that the ordinary Veneziano amplitude, which is a special function, can be expressed as product of all inverse p-adic counterparts, which are elementary functions. This is here a consequence of the Gel'fand-Graev-Tate beta functions and it is not a general property of string scattering amplitudes. In a general case product of string amplitudes will not be a constant but a function of kinematical variables.

Note that there is another interpretation of product expression (8) which is related to scattering amplitude of two adelic strings, but it seems to be incorrect. Namely, recall that ordinary and p-adic strings have real and p-adic world-sheets, respectively. Then under an adelic string we should understand a string which has an adelic world-sheet. However, it has not been obtained so far the above scattering amplitude for two open scalar strings with their adelic world-sheets. Thus, an adelic string with adelic world-sheet is not well founded. But p-adic strings with p-adic world-sheet are well defined, and the product of their scattering amplitudes has a useful meaning. For an approach to adelic strings with adelic ambient space, see [18].

4. On Zeta Strings

We see that p-adic amplitudes (3) have the same form for every p, and difference is only in prime number p and coupling constant g_p^2 (when $g_p^2 \neq 1$). In (7) it is presented product of scattering amplitudes for all primes p and result is expressed through the ratios of the Riemann zeta function. It gives rise to attempt to treat all p-adic strings together as a whole, which we call zeta strings. To this end, our intention is to construct an appropriate Lagrangian to describe this p-adic sector. It is obvious that such Lagrangian should contain Riemann zeta function, which has d'Alembertian $\Box$ in its argument. Thus we have to look for possible constructions of a Lagrangian which contains the Riemann zeta function and has its origin in p-adic Lagrangian (4). We have found and considered two approaches: additive and multiplicative.

4.1. Additive approach

Prime number p in (4) can be replaced by any natural number $n \geq 2$ and consequences also make sense. Now we want to introduce a Lagrangian which incorporates all the above Lagrangians (4), with p replaced by $n \in \mathbb{N}$. To this end, we take the sum of all Lagrangians $\mathcal{L}_n$ in the form

$$L = \sum_{n=1}^{+\infty} C_n \, \mathcal{L}_n = \sum_{n=1}^{+\infty} C_n \frac{m^D}{g_n^2} \frac{n^2}{n-1} \Big[-\frac{1}{2} \, \phi \, n^{-\frac{\Box}{2m^2}} \, \phi + \frac{1}{n+1} \, \phi^{n+1} \Big] , \qquad (9)$$

whose explicit realization depends on particular choice of coefficients C_n and coupling constants g_n. To avoid a divergence in $1/(n-1)$ when $n = 1$ we take that C_n/g_n^2 is proportional to $n - 1$. Here we consider some cases when coefficients C_n are proportional to $n - 1$, while coupling constants g_n do not depend on n, i.e. $g_n = g$. In fact, according to (8), in this case we have $g_n^2 = g^2 = 1$. Another possibility is that C_n is not proportional to $n - 1$, but $g_n^2 = \frac{n^2}{n^2-1}$ and then $\prod_p g_p^2 = \zeta(2) = \frac{\pi^2}{6}$, what is consistent with (8). To differ this new field from a particular p-adic one, we use notation ϕ instead of φ.

We have considered five cases for coefficients C_n in (9): (i) $C_n = \frac{n-1}{n^{2+h}}$, where h is a real parameter; (ii) $C_n = \frac{n^2-1}{n^2}$; (iii) $C_n = \mu(n) \, \frac{n-1}{n^2}$, and (iv) $C_n = -\mu(n) \, \frac{n^2-1}{n^2}$, where $\mu(n)$ is the Möbius function; and (v) $C_n = (-1)^{n-1} \, \frac{n^2-1}{n^2}$.

Case (i) was considered in [19, 20]. Obtained Lagrangian is

$$L_h = \frac{m^D}{g^2} \Big[-\frac{1}{2} \, \phi \, \zeta\Big(\frac{\Box}{2\,m^2} + h\Big) \, \phi + \mathcal{AC} \sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1} \, \phi^{n+1} \Big] , \qquad (10)$$

where $\mathcal{AC}$ denotes analytic continuation.

Case (ii) was investigated in [21] and the corresponding Lagrangian is

$$L = \frac{m^D}{g^2} \Big[-\frac{1}{2} \, \phi \Big\{ \zeta\Big(\frac{\Box}{2\,m^2} - 1\Big) + \zeta\Big(\frac{\Box}{2\,m^2}\Big) \Big\} \, \phi + \frac{\phi^2}{1-\phi} \Big] . \qquad (11)$$

Cases with the Möbius function $\mu(n)$ are presented in [22] and [23]. Recall that the Möbius function is defined for all positive integers and has values $1, 0, -1$

depending on factorization of n into prime numbers p. Its explicit definition is

$$\mu(n) = \begin{cases} 0, & n = p^2 m\,, \\ (-1)^k, & n = p_1 p_2 \cdots p_k,\ p_i \neq p_j\,, \\ 1, & n = 1,\ (k = 0)\,. \end{cases} \tag{12}$$

Möbius function is related to the inverse Riemann zeta function as follows:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}, \quad s = \sigma + i\tau, \quad \sigma > 1\,. \tag{13}$$

The corresponding Lagrangian for $C_n = \mu(n)\, \frac{n-1}{n^2}$ is

$$L = \frac{m^D}{g^2}\Big[-\frac{1}{2}\,\phi\, \frac{1}{\zeta\Big(\frac{\Box}{2m^2}\Big)}\,\phi + \int_0^\phi \mathcal{M}(\phi)\, d\phi \Big], \tag{14}$$

where $\mathcal{M}(\phi) = \sum_{n=1}^{+\infty} \mu(n)\, \phi^n = \phi - \phi^2 - \phi^3 - \phi^5 + \phi^6 - \phi^7 + \phi^{10} - \phi^{11} - \dots$.

When $C_n = -\mu(n)\, \frac{n^2-1}{n^2}$ then the Lagrangian is

$$L = \frac{m^D}{g^2}\,\Big\{ \frac{1}{2}\,\phi \Big[\frac{1}{\zeta\Big(\frac{\Box}{2m^2} - 1\Big)} + \frac{1}{\zeta\Big(\frac{\Box}{2m^2}\Big)} \Big]\,\phi - \phi^2\, F(\phi) \Big\}\,, \tag{15}$$

where $F(\phi) = \sum_{n=1}^{+\infty} \mu(n) \phi^{n-1} = 1 - \phi - \phi^2 - \phi^4 + \dots$.

The case (v) with $C_n = (-1)^{n-1}\, \frac{n^2-1}{n^2}$ was introduced recently in [24]. This choice of coefficients C_n is similar to the above case (ii) and distinction is in the sign $(-1)^{n-1}$. Recall that

$$\sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1}{n^s} = (1 - 2^{1-s})\,\zeta(s), \quad s = \sigma + i\tau\,, \quad \sigma > 0\,, \tag{16}$$

which has analytic continuation to the entire complex s plane without singularities. At point $s = 1$, one has $\lim_{s\to 1}(1 - 2^{1-s})\,\zeta(s) = \sum_{n=1}^{+\infty}(-1)^{n-1}\frac{1}{n} = \log 2$. Applying (16) to (9) and using analytic continuation one obtains

$$\begin{aligned} L = -\,\frac{m^D}{g^2}\Big[\frac{1}{2}\,\phi \Big\{ \Big(1 - 2^{2-\frac{\Box}{2m^2}}\Big)\,\zeta\Big(\frac{\Box}{2\,m^2} - 1\Big) \\ + \Big(1 - 2^{1-\frac{\Box}{2m^2}}\Big)\,\zeta\Big(\frac{\Box}{2\,m^2}\Big) \Big\}\,\phi - \frac{\phi^2}{1+\phi} \Big]\,. \end{aligned} \tag{17}$$

Some new Lagrangians can be easily constructed starting from the following expressions for the Riemann zeta function related to important functions in number theory (see, e.g. [25]):

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}\,, \quad s = \sigma + i\tau,\ \sigma > 2\,, \tag{18}$$

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s}\,, \quad s = \sigma + i\tau,\ \sigma > 1\,, \tag{19}$$

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}\,, \quad s = \sigma + i\tau,\ \sigma > 1\,, \tag{20}$$

where $\varphi(n)$, $\mu(n)$ and $\lambda(n)$ are Euler's phi function, Möbius and Liouville function, respectively. Recall that $\varphi(n)$ is equal to the number of positive numbers less than n (and equal to n) that are coprime to n. The Liouville function is $\lambda(n) = (-1)^{\Omega(n)}$, where

$$\Omega(n) = \begin{cases} 0, & n = 1\,, \\ k, & n = p_1 p_2 \cdots p_k\,, \end{cases} \tag{21}$$

i.e. $\Omega(n)$ is the number of prime factors of n, counted with their multiplicity.

4.2. Multiplicative approach

In the multiplicative approach the Riemann zeta function emerges through its product form (6). Our starting point is again p-adic Lagrangian (4) with $g_p^2 = g^2 = 1$. It is useful to rewrite (4), first in the form,

$$\mathcal{L}_p = \frac{m^D}{g^2}\,\frac{p^2}{p^2-1}\Big\{ -\frac{1}{2}\,\varphi\left[p^{-\frac{\Box}{2m^2}+1} + p^{-\frac{\Box}{2m^2}}\right]\varphi + \varphi^{p+1}\Big\} \tag{22}$$

and then, after addition and substraction of φ^2, as

$$\begin{aligned}\mathcal{L}_p =& \frac{m^D}{g^2}\,\frac{p^2}{p^2-1}\Big\{\frac{1}{2}\,\varphi\left[\Big(1 - p^{-\frac{\Box}{2m^2}+1}\Big) + \Big(1 - p^{-\frac{\Box}{2m^2}}\Big)\right]\varphi \\ &- \varphi^2\Big(1 - \varphi^{p-1}\Big)\Big\}\,. \end{aligned} \tag{23}$$

Taking products

$$\prod_p \frac{1}{1-p^{-2}}\,,\ \prod_p (1 - p^{-\frac{\Box}{2m^2}+1})\,,\ \prod_p (1 - p^{-\frac{\Box}{2m^2}})\,,\ \prod_p (1 - \varphi^{p-1}) \tag{24}$$

in (23) at the relevant places one obtains Lagrangian

$$\mathcal{L} = \frac{m^D}{g^2}\,\zeta(2)\,\Big\{\frac{1}{2}\,\phi\Big[\frac{1}{\zeta\Big(\frac{\Box}{2m^2}-1\Big)} + \frac{1}{\zeta\Big(\frac{\Box}{2m^2}\Big)}\Big]\,\phi - \phi^2\,\Phi(\phi)\Big\}\,, \qquad (25)$$

where $\Phi(\phi) = \prod_p \Big(1-\phi^{p-1}\Big) = 1-\phi-\phi^2+\phi^3-\phi^4+....$ It is worth noting that from Lagrangian (25) one can easily reproduce its p-adic ingredient (23). It is obvious that Lagrangian (25) is similar to this one in (15). These two Lagrangians describe the same field theory in the week field approximation. Lagrangian (25) was introduced and considered in [23].

5. Concluding remarks

In the previous sections we have presented only some Lagrangians which are candidates for description of the p-adic sector of open scalar strings, which we call open scalar zeta string. They contain the Riemann zeta function and they are also starting points for interesting examples of what we call zeta scalar field theory. The corresponding potentials, which are $V(\phi) = -L(\Box = 0)$, and equations of motions are considered in cited references. All these zeta field theory models contain tachyons.

One of the most interesting of the above Lagrangians is presented in (17). Unlike other Lagrangians, this one has no singularity with respect to the d'Alembertian $\Box$ and it enables to apply easier its pseudodifferential treatment. This analyticity of the Lagrangian should be also useful in its application to nonlocal cosmology, which uses linearization procedure (see, e.g. [26] and references therein).

We would like also to point out Lagrangians (15) and (25), since they are mutually very similar. These Lagrangians describe the same model at the weak field approximation, although they are constructed using rather different approaches.

An interesting approach towards foundation of a field theory and cosmology based on the Riemann zeta function was proposed in [27].

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