Invariant Differential Operators for Non-Compact Lie Groups: the $E_{6(-14)}$ case

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Abstract

In the present paper we continue the project of systematic explicit construction of invariant differential operators. On the example of the non-compact exceptional group $E_{6(-14)}$ we give the multiplets of indecomposable elementary representations. This includes the data for all relevant invariant differential operators.

1. Introduction

Invariant differential operators play very important role in the description of physical symmetries - starting from the early occurrences in the Maxwell, d'Allembert, Dirac, equations, (for more examples cf., e.g., [1]), to the latest applications of (super-)differential operators in conformal field theory, supergravity and string theory, (for a recent review, cf. e.g., [2]). Thus, it is important for the applications in physics to study systematically such operators.

In a recent paper [3] we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

In the present paper we focus on one particular group $E_{6(-14)}$, which is very interesting for at least two reasons. First of all exceptional groups are

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still not much studied and used, cf. though [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Furthermore, $E_{6(-14)}$ is one of two exceptional non-compact groups that have highest/lowest weight representations.[1]

In our further plans it shall be very useful that (as in [3]) we follow a procedure in representation theory in which intertwining differential operators appear canonically [15] and which procedure has been generalized to the supersymmetry setting [16, 17] and to quantum groups [18]. (For more references, cf. [3].)

The present paper is organized a follows. In section 2 we give the preliminaries, actually recalling and adapting facts from [3]. In Section 3 we specialize to the $E_{6(-14)}$ case. In Section 4 we present our results on the multiplet classification of the representations and intertwining differential operators between them.

2. Preliminaries

Let G be a semisimple non-compact Lie group, and K a maximal compact subgroup of G. Then we have an Iwasawa decomposition $G = KA_0N_0$, where A_0 is abelian simply connected vector subgroup of G, N_0 is a nilpotent simply connected subgroup of G preserved by the action of A_0. Further, let M_0 be the centralizer of A_0 in K. Then the subgroup $P_0 = M_0A_0N_0$ is a minimal parabolic subgroup of G. A parabolic subgroup $P = M'A'N'$ is any subgroup of G (including G itself) which contains a minimal parabolic subgroup.[2]

The importance of the parabolic subgroups comes from the fact that the representations induced from them generate all (admissible) irreducible representations of G [20]. For the classification of all irreducible representations it is enough to use only the so-called *cuspidal* parabolic subgroups $P = M'A'N'$, singled out by the condition that $\operatorname{rank} M' = \operatorname{rank} M' \cap K$ [21, 22], so that M' has discrete series representations [23]. However, often induction from non-cuspidal parabolics is also convenient, cf. [24, 3, 25].

Let ν be a (non-unitary) character of A', $\nu \in \mathcal{A}'^*$, let μ fix an irreducible representation D^μ of M' on a vector space V_μ.

We call the induced representation $\chi = \mathrm{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G [26]. (These are called *generalized principal series representations* (or *limits thereof*) in [27].) Their spaces of functions are:

$$\mathcal{C}_\chi = \{\mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(gman) = e^{-\nu(H)} \cdot D^\mu(m^{-1})\,\mathcal{F}(g)\} \quad (2.1)$$

where $a = \exp(H) \in A'$, $H \in \mathcal{A}'$, $m \in M'$, $n \in N'$. The representation

[1] The other one is $E_{7(-25)}$ which we also plan to consider.

[2] The number of non-conjugate parabolic subgroups is 2^r, where $r = \operatorname{rank} A$, cf., e.g., [19].

action is the *left* regular action:

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') \;=\; \mathcal{F}(g^{-1}g') \;, \quad g,g' \in G \;. \tag{2.2}$$

For our purposes we need to restrict to *maximal* parabolic subgroups P, (so that $\operatorname{rank} A' = 1$), that may not be cuspidal. For the representations that we consider the character ν is parameterized by a real number d, called the conformal weight or energy.

Further, let μ fix a discrete series representation D^μ of M' on the Hilbert space V_μ, or the so-called limit of a discrete series representation (cf. [27]). Actually, instead of the discrete series we can use the finite-dimensional (non-unitary) representation of M' with the same Casimirs.

An important ingredient in our considerations are the *highest/lowest weight representations* of $\mathcal{G}$. These can be realized as (factor-modules of) Verma modules V^Λ over $\mathcal{G}^{\mathbb{C}}$, where $\Lambda \in (\mathcal{H}^{\mathbb{C}})^*$, $\mathcal{H}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathcal{G}^{\mathbb{C}}$, weight $\Lambda = \Lambda(\chi)$ is determined uniquely from χ [15]. In this setting we can consider also unitarity, which here means positivity w.r.t. the Shapovalov form in which the conjugation is the one singling out $\mathcal{G}$ from $\mathcal{G}^{\mathbb{C}}$.

Actually, since our ERs may be induced from finite-dimensional representations of $\mathcal{M}'$ (or their limits) the Verma modules are always reducible. Thus, it is more convenient to use *generalized Verma modules* $\tilde{V}^\Lambda$ such that the role of the highest/lowest weight vector v_0 is taken by the (finite-dimensional) space $V_\mu\, v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight d. Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential.

One main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called *multiplets* [28, 15]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible ERs and the lines between the vertices correspond to intertwining operators.[3] The explicit parametrization of the multiplets and of their ERs is important for understanding of the situation.

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consists of the pair (β, m), where β is a (non-compact) positive root of $\mathcal{G}^{\mathbb{C}}$, $m \in \mathbb{N}$, such that the BGG [29] Verma module reducibility condition (for highest weight modules) is fulfilled:

$$(\Lambda + \rho, \beta^\vee) \;=\; m \;, \quad \beta^\vee \equiv 2\beta/(\beta,\beta) \;. \tag{2.3}$$

[3] For simplicity only the operators which are not compositions of other operators are depicted.

When (2.3) holds then the Verma module with shifted weight $V^{\Lambda-m\beta}$ (or $\tilde{V}^{\Lambda-m\beta}$ for GVM and β non-compact) is embedded in the Verma module V^{Λ} (or $\tilde{V}^{\Lambda}$). This embedding is realized by a singular vector v_s determined by a polynomial $\mathcal{P}_{m,\beta}(\mathcal{G}^-)$ in the universal enveloping algebra $(U(\mathcal{G}_-))\, v_0$, $\mathcal{G}^-$ is the subalgebra of $\mathcal{G}^{\mathbb{C}}$ generated by the negative root generators [30]. More explicitly, [15], $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, v_0$ (or $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, V_\mu\, v_0$ for GVMs).[4] Then there exists [15] an intertwining differential operator

$$\mathcal{D}_{m,\beta} \ :\ \mathcal{C}_{\chi(\Lambda)} \ \longrightarrow\ \mathcal{C}_{\chi(\Lambda-m\beta)} \tag{2.4}$$

given explicitly by:

$$\mathcal{D}_{m,\beta} \ =\ \mathcal{P}_{m,\beta}(\widehat{\mathcal{G}^-}) \tag{2.5}$$

where $\widehat{\mathcal{G}^-}$ denotes the *right* action on the functions $\mathcal{F}$, cf. (2.1).

3. The non-compact Lie algebra $E_{6(-14)}$

Let $\mathcal{G} = E_{6(-14)}$, [32]. This non-compact Lie algebra is also denoted as EIII [33], or E_6''' [34]. The maximal compact subalgebra is $\mathcal{K} \cong so(10) \oplus so(2)$, $\dim_{\mathbb{R}} \mathcal{P} = 32$, $\dim_{\mathbb{R}} \mathcal{N}^{\pm} = 30$. This real form has discrete series representations and highest/lowest weight representations. The split rank is equal to 2, while $\mathcal{M}_0 \cong so(6) \oplus so(2)$.

The Satake diagram [35] is:

$$\begin{array}{c} \qquad\qquad\quad \circ\,\alpha_2 \\ \qquad\qquad\quad | \\ \underbrace{\underset{\alpha_1}{\circ} \;---\; \underset{\alpha_3}{\bullet} \;---\; \underset{\alpha_4}{\bullet} \;---\; \underset{\alpha_5}{\bullet} \;---\; \underset{\alpha_6}{\circ}} \end{array} \tag{3.1}$$

Thus, the reduced root system is presented by a Dynkin-Satake diagram looking like the B_2 Dynkin diagram with simple roots λ_1, λ_2, but the long roots (including λ_1) have multiplicity 6, while the short roots (including λ_2) have multiplicity 8, and there are also the roots 2λ of multiplicity 1, where λ is any short root. It is obtained from (3.1) by dropping the black nodes, (they give rise to $\mathcal{M}_0$), identifying α_1 and α_6 and mapping them to λ_2, while the root α_2 is mapped to λ_1.

The non-minimal parabolic subalgebras are given by:

$$\begin{aligned} \mathcal{M} &= su(5,1)\ , \qquad \dim \mathcal{N}^{\pm} \ =\ 21\ , \qquad \dim \mathcal{A} \ =\ 1 \\ \mathcal{M}' &= so(7,1) \oplus so(2)\ , \qquad \dim \mathcal{N}'^{\pm} \ =\ 24\ , \qquad \dim \mathcal{A}' \ =\ 1 \end{aligned} \tag{3.2}$$

Both are maximal and the first is cuspidal.

[4] For explicit expressions for singular vectors we refer to [31].

We shall induce our representations from the first parabolic $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$.

We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\chi = \{ n_1 , n_3 , n_4 , n_5 , n_6 ; c \} , \quad c = d - \tfrac{11}{2} , \tag{3.3}$$

where the last entry of χ labels the characters of $\mathcal{A}$, and the first five entries are labels of the discrete series of $\mathcal{M}$, then $n_j \in I\!N$, or of limits of discrete series, when some of n_j are zero. (When $n_j \in I\!N$ the first five entries label also the finite-dimensional nonunitary irreps of $\mathcal{M}$, and the finite-dimensional unitary irreps of $su(6)$).

The reason to use the parameter c instead of d is that the parametrization of the ERs in the multiplets is given in a simpler way.

Further, we need the root system of the complex algebra E_6. With Dynkin diagram enumerating the simple roots α_i as in (3.1), the positive roots are: first there are the 15 positive roots of $sl(6)$ with simple roots $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6$, then there are the following 21 roots:

$$\begin{aligned}
&\alpha_2 , \quad \alpha_2+\alpha_4 , \quad \alpha_2+\alpha_4+\alpha_3 , \quad \alpha_2+\alpha_4+\alpha_5 , \\
&\alpha_2+\alpha_4+\alpha_3+\alpha_5 , \quad \alpha_2+\alpha_4+\alpha_3+\alpha_1 , \quad \alpha_2+\alpha_4+\alpha_5+\alpha_6 , \\
&\alpha_2+\alpha_4+\alpha_3+\alpha_5+\alpha_1 , \quad \alpha_2+\alpha_4+\alpha_3+\alpha_5+\alpha_6 , \quad \alpha_2+2\alpha_4+\alpha_3+\alpha_5, \\
&\alpha_2+\alpha_4+\alpha_3+\alpha_5+\alpha_1+\alpha_6 , \quad \alpha_2+2\alpha_4+\alpha_3+\alpha_5+\alpha_1 , \\
&\alpha_2+2\alpha_4+\alpha_3+\alpha_5+\alpha_6 , \quad \alpha_2+2\alpha_4+\alpha_3+\alpha_5+\alpha_1+\alpha_6 , \\
&\alpha_2+2\alpha_4+2\alpha_3+\alpha_5+\alpha_1 , \quad \alpha_2+2\alpha_4+\alpha_3+2\alpha_5+\alpha_6 , \\
&\alpha_2+2\alpha_4+2\alpha_3+\alpha_5+\alpha_1+\alpha_6 , \quad \alpha_2+2\alpha_4+\alpha_3+2\alpha_5+\alpha_1+\alpha_6 , \\
&\alpha_2+2\alpha_4+2\alpha_3+2\alpha_5+\alpha_1+\alpha_6 , \\
&\alpha_2+3\alpha_4+2\alpha_3+2\alpha_5+\alpha_1+\alpha_6 , \\
&2\alpha_2+3\alpha_4+2\alpha_3+2\alpha_5+\alpha_1+\alpha_6 \equiv \tilde{\alpha} ,
\end{aligned} \tag{3.4}$$

where $\tilde{\alpha}$ is the highest root of the E_6 root system.

Relative to our parabolic subalgebra, the roots in (3.4) are non-compact, while the rest are compact. The differential intertwining operators that give the multiplets correspond to the noncompact roots, and since we shall use the latter extensively, we introduce more compact notation for them. Namely, the nonsimple roots in (3.4) will be denoted in a self-explanatory way as follows:

$$\begin{aligned}
\alpha_{ij} &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_j , \quad \alpha_{i,j} = \alpha_i + \alpha_j , \quad i<j , \\
\alpha_{ij,k} &= \alpha_{k,ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j + \alpha_k , \quad i<j , \\
\alpha_{ij,km} &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_j + \alpha_k + \alpha_{k+1} + \cdots + \alpha_m , \quad i<j , \quad k<m, \\
\alpha_{ij,km,4} &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_j + \alpha_k + \alpha_{k+1} + \cdots + \alpha_m + \alpha_4 , \\
&\quad i<j , \quad k<m ,
\end{aligned} \tag{3.5}$$

i.e., the roots (3.4) will be written as:

$$\begin{aligned}
&\alpha_2 , \quad \alpha_{2,4} , \quad \alpha_{24} , \quad \alpha_{2,45} , \quad \alpha_{25} , \quad \alpha_{14} , \quad \alpha_{2,46} , \\
&\alpha_{15} , \quad \alpha_{26} , \quad \alpha_{25,4} , \quad \alpha_{16} , \quad \alpha_{15,4} , \quad \alpha_{26,4} ,
\end{aligned} \tag{3.6}$$

$$\begin{array}{l} \alpha_{16,4}\ ,\quad \alpha_{15,34}\ ,\quad \alpha_{26,45}\ ,\quad \alpha_{16,34}\ ,\quad \alpha_{16,45}\ , \\ \alpha_{16,35}\ ,\quad \alpha_{16,35,4}\ ,\quad \alpha_{16,25,4}\ =\ \tilde{\alpha}\ . \end{array}$$

Further, we give the correspondence between the signatures χ and the highest weight Λ. The connection is through the Dynkin labels:

$$m_i \ \equiv\ (\Lambda+\rho, \alpha_i^\vee)\ =\ (\Lambda+\rho, \alpha_i)\ ,\quad i=1,\ldots,6, \tag{3.7}$$

where $\Lambda=\Lambda(\chi)$, ρ is half the sum of the positive roots of $\mathcal{G}^{I\!\!C}$, α_i denotes the simple roots of $\mathcal{G}^{I\!\!C}$. The explicit connection is:

$$n_i = m_i\ ,\quad -c\ =\ \tfrac{1}{2} n_{\tilde{\alpha}}\ =\ \tfrac{1}{2}(n_1+2n_2+2n_3+3n_4+2n_5+n_6) \tag{3.8}$$

We shall use also the so-called Harish-Chandra parameters:

$$m_\beta \equiv (\Lambda+\rho, \beta)\ , \tag{3.9}$$

where β is any positive root of $\mathcal{G}^{I\!\!C}$. These parameters are redundant, since obviously they are expressed in terms of the Dynkin labels, however, some statements are best formulated in their terms.[5]

There are several types of multiplets: the main type, which contains maximal number of ERs/GVMs, the finite-dimensional and the discrete series representations, and some reduced types of multiplets.

In the next Section we give the classification of all multiplets that are physically relevant.

4. Multiplets

4.1. Main type of multiplets

The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of $\mathcal{G}$, i.e., they will be labelled by the six positive Dynkin labels $m_i \in I\!\!N$. It turns out that each such multiplet contains 70 ERs/GVMs whose signatures can be given in the following pair-wise manner:

$$\begin{aligned} \chi_0^\pm\ &=\ \{\,m_1\,,m_3\,,m_4\,,m_5\,,m_6\,;\,\pm\tfrac{1}{2}m_{\tilde{\alpha}}\,\}\ , \\ \chi_a^\pm\ &=\ \{\,m_1\,,m_3\,,m_{2,4}\,,m_5\,,m_6\,;\,\pm\tfrac{1}{2}(m_{\tilde{\alpha}}-m_2)\,\}\ , \\ \chi_b^\pm\ &=\ \{\,m_1\,,m_{34}\,,m_2\,,m_{45}\,,m_6\,;\,\pm\tfrac{1}{2}(m_{\tilde{\alpha}}-m_{2,4})\,\}\ , \\ \chi_c^\pm\ &=\ \{\,m_{1,3}\,,m_4\,,m_2\,,m_{35}\,,m_6\,;\,\pm\tfrac{1}{2}(m_{\tilde{\alpha}}-m_{24})\,\}\ , \\ \chi_{c'}^\pm\ &=\ \{\,m_1\,,m_{35}\,,m_2\,,m_4\,,m_{56}\,;\,\pm\tfrac{1}{2}(m_{\tilde{\alpha}}-m_{2,45})\,\}\ , \end{aligned} \tag{4.1}$$

[5]Clearly, both the Dynkin and Harish-Chandra labels have their origin in the BGG reducibility condition (2.3).

$$\begin{aligned}
\chi_d^\pm &= \{\, m_3\,, m_4\,, m_2\,, m_{1,35}\,, m_6\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{14})\,\}\ ,\\
\chi_{d'}^\pm &= \{\, m_1\,, m_{36}\,, m_2\,, m_4\,, m_5\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{2,46})\,\}\ ,\\
\chi_{\tilde d}^\pm &= \{\, m_{1,3}\,, m_{45}\,, m_2\,, m_{34}\,, m_{56}\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{25})\,\}\ ,\\
\chi_e^\pm &= \{\, m_3\,, m_{45}\,, m_2\,, m_{1,34}\,, m_{56}\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{15})\,\}\ ,\\
\chi_{e'}^\pm &= \{\, m_{1,3}\,, m_{46}\,, m_2\,, m_{34}\,, m_5\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{26})\,\}\ ,\\
\chi_{\tilde e}^\pm &= \{\, m_3\,, m_{46}\,, m_2\,, m_{1,34}\,, m_5\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{16})\,\}\ ,\\
\chi_f^\pm &= \{\, m_{34}\,, m_5\,, m_{2,4}\,, m_{1,3}\,, m_{46}\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{15,4})\,\}\ ,\\
\chi_{f'}^\pm &= \{\, m_{1,34}\,, m_{56}\,, m_{2,4}\,, m_3\,, m_{45}\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{26,4})\,\}\ ,\\
\chi_{\tilde f}^\pm &= \{\, m_{1,34}\,, m_5\,, m_{2,4}\,, m_3\,, m_{46}\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{25,4})\,\}\ ,\\
\chi_{f^o}^\pm &= \{\, m_4\,, m_5\,, m_{24}\,, m_1\,, m_{36}\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{15,34})\,\}\ ,\\
\chi_{f''}^\pm &= \{\, m_{1,35}\,, m_6\,, m_{2,45}\,, m_3\,, m_4\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{26,45})\,\}\ ,\\
\chi_g^\pm &= \{\, m_{24}\,, m_5\,, m_4\,, m_{1,3}\,, m_{2,46}\,;\, \pm\tfrac{1}{2}m_{36}\,\}\ ,\\
\chi_{g'}^\pm &= \{\, m_{14}\,, m_{56}\,, m_4\,, m_3\,, m_{2,45}\,;\, \pm\tfrac{1}{2}m_{1,35}\,\}\ ,\\
\chi_{\tilde g}^\pm &= \{\, m_{14}\,, m_5\,, m_4\,, m_3\,, m_{2,46}\,;\, \pm\tfrac{1}{2}m_{1,36}\,\}\ ,\\
\chi_{g^o}^\pm &= \{\, m_{2,4}\,, m_5\,, m_{34}\,, m_1\,, m_{26}\,;\, \pm\tfrac{1}{2}m_{46}\,\}\ ,\\
\chi_{g''}^\pm &= \{\, m_{15}\,, m_6\,, m_{45}\,, m_3\,, m_{2,4}\,;\, \pm\tfrac{1}{2}m_{1,34}\,\}\ ,\\
\chi_h^\pm &= \{\, m_4\,, m_{56}\,, m_{24}\,, m_1\,, m_{35}\,;\, \pm\tfrac{1}{2}m_{2,45}\,\}\ ,\\
\chi_{h'}^\pm &= \{\, m_{35}\,, m_6\,, m_{2,45}\,, m_{1,3}\,, m_4\,;\, \pm\tfrac{1}{2}m_{24}\,\}\ ,\\
\chi_{\tilde h}^\pm &= \{\, m_{34}\,, m_{56}\,, m_{2,4}\,, m_{1,3}\,, m_{45}\,;\, \pm\tfrac{1}{2}m_{25}\,\}\ ,\\
\chi_{\hat h}^\pm &= \{\, m_{45}\,, m_6\,, m_{25}\,, m_1\,, m_{34}\,;\, \pm\tfrac{1}{2}m_{2,4}\,\}\ ,\\
\chi_{\hat k}^\pm &= \{\, m_{24}\,, m_{56}\,, m_4\,, m_{1,3}\,, m_{2,45}\,;\, \pm\tfrac{1}{2}m_{35}\,\}\ ,\\
\chi_j^\pm &= \{\, m_{2,4}\,, m_{56}\,, m_{34}\,, m_1\,, m_{25}\,;\, \pm\tfrac{1}{2}m_{45}\,\}\ ,\\
\chi_{j'}^\pm &= \{\, m_{25}\,, m_6\,, m_{45}\,, m_{1,3}\,, m_{2,4}\,;\, \pm\tfrac{1}{2}m_{34}\,\}\ ,\\
\chi_{j^o}^\pm &= \{\, m_2\,, m_{45}\,, m_3\,, m_1\,, m_{26,4}\,;\, \pm\tfrac{1}{2}m_{56}\,\}\ ,\\
\chi_{j''}^\pm &= \{\, m_{15,4}\,, m_6\,, m_5\,, m_{34}\,, m_2\,;\, \pm\tfrac{1}{2}m_{1,3}\,\}\ ,\\
\chi_k^\pm &= \{\, m_2\,, m_{46}\,, m_3\,, m_1\,, m_{25,4}\,;\, \pm\tfrac{1}{2}m_5\,\}\ ,\\
\chi_{k'}^\pm &= \{\, m_{25,4}\,, m_6\,, m_5\,, m_{1,34}\,, m_2\,;\, \pm\tfrac{1}{2}m_3\,\}\ ,\\
\chi_{\tilde k}^\pm &= \{\, m_{2,45}\,, m_6\,, m_{35}\,, m_1\,, m_{24}\,;\, \pm\tfrac{1}{2}m_4\,\}\ ,\\
\chi_{k^o}^\pm &= \{\, m_2\,, m_4\,, m_3\,, m_1\,, m_{26,45}\,;\, \pm\tfrac{1}{2}m_6\,\}\ ,\\
\chi_{k''}^\pm &= \{\, m_{15,34}\,, m_6\,, m_5\,, m_4\,, m_2\,;\, \pm\tfrac{1}{2}m_1\,\}\ ,
\end{aligned}$$

where we have used for the numbers $m_\beta = (\Lambda(\chi) + \rho, \beta)$ the same compact notation as in (3.5) for the roots β.

The ERs in the multiplet are related by intertwining integral and differential operators. The integral operators were introduced by Knapp and Stein [36]. The relevant fact here is that the above pairs are related by Knapp-Stein integral operators, such that the weights are reflected by the maximal root:

$$s_{\tilde{\alpha}} \cdot \Lambda(\chi^{\pm}) = \Lambda(\chi^{\mp}) \tag{4.2}$$

These operators intertwining the pairs will be denoted by:

$$G^{\pm} \ : \ \mathcal{C}_{\chi^{\mp}} \longrightarrow \mathcal{C}_{\chi^{\pm}} \tag{4.3}$$

Matters are arranged so that in every multiplet only the ER with signature χ_0^- contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional irrep of E_6 with signature $\{m_1 , \ldots , m_6\}$. The subspace $\mathcal{E}$ is annihilated by the operator G^+, and is the image of the operator G^-. When all $m_i = 1$ then $\dim \mathcal{E} = 1$, and in that case $\mathcal{E}$ is also the trivial one-dimensional UIR of the whole algebra $E_{6(-14)}$. Furthermore in that case the conformal weight is zero: $d = \frac{11}{2} + c = \frac{1}{2}(11 - m_{\tilde{\alpha}|_{m_i=1}}) = 0$.

Analogously, in every multiplet only the ER with signature χ_0^+ contains holomorphic discrete series representation. This is guaranteed by the fact that for this ER all Harish-Chandra parameters for non-compact roots are negative, i.e., $n_\alpha < 0$, for α from (3.4). [The last fact can be easily checked from the signatures (4.1).] The conformal weight has the restriction $d = \frac{1}{2}(11 + m_{\tilde{\alpha}}) \geq 11$.

In fact, this ER contains also the conjugate anti-holomorphic discrete series. The direct sum of the holomorphic and the antiholomorphic representations are realized in an invariant subspace $\mathcal{D}$ of the ER χ_0^+. That subspace is annihilated by the operator G^-, and is the image of the operator G^+.

Note that the corresponding lowest weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series.

The intertwining differential operators correspond to non-compact positive roots of the root system of E_6, cf. [15], i.e., in the current context, the roots given in (3.4), (or in more compact notation, (3.6)).

The multiplets are given explicitly in Fig. 1. Each intertwining differential operator is represented by an arrow accompanied by a symbol $i_{j\ldots k}$ encoding the root $\beta_{j\ldots k}$ and the number $m_{\beta_{j\ldots k}}$ which is involved in the BGG criterion. This notation is used to save space, but it can be used due to the fact that only intertwining differential operators which are non-composite are displayed, and that the data β, m_β, which is involved in the embedding $V^\Lambda \longrightarrow V^{\Lambda - m_\beta,\beta}$ turns out to involve only the m_i corresponding to simple roots, i.e., for each β, m_β there exists $i = i(\beta, m_\beta, \Lambda) \in \{1, \ldots, 6\}$, such that $m_\beta = m_i$. Hence the data $\beta_{j\ldots k}, m_{\beta_{j\ldots k}}$ is represented by $i_{j\ldots k}$ on the arrows.

Note that there are five cases when the embeddings correspond to the highest root $\tilde{\alpha}$: $V^{\Lambda^-} \longrightarrow V^{\Lambda^+}$, $\Lambda^+ = \Lambda^- - m_{\tilde{\alpha}}\,\tilde{\alpha}$. In these five cases the weights are denoted as: $\Lambda^\pm_{k''}$, $\Lambda^\pm_{k'}$, $\Lambda^\pm_{\tilde{k}}$, $\Lambda^\pm_{\hat{k}}$, $\Lambda^\pm_{k^o}$, then we have: $m_{\tilde{\alpha}} = m_1, m_3, m_4, m_5, m_6$, resp. We recall that Knapp-Stein operators G^+ intertwine the corresponding ERs $\mathcal{T}^-_\chi$ and $\mathcal{T}^+_\chi$, cf. (4.3). In the above five cases the Knapp-Stein operators G^+ degenerate to differential operators. The latter phenomenon is given explicitly for the case of the anti-de-Sitter algebra $so(3,2)$ in [37].

Note that the figure has the standard E_6 symmetry, namely, conjugation exchanging indices $1 \longleftrightarrow 6$, $3 \longleftrightarrow 5$. This conjugation will be used also below when we describe the reduced embedding diagrams.

In the next Subsections we shall consider the reduced types of multiplets.

4.2. Main types of reduced multiplets

4.2.1. Reduced type R2 multiplets

The multiplets of reduced type R2 contain 50 ERs/GVMs and may be obtained formally from the main type by setting $m_2 = 0$. Their signatures are given explicitly by:

$$\begin{aligned}
\chi^\pm_a &= \{\, m_1\,, m_3\,, m_4\,, m_5\,, m_6\,;\ \pm\tfrac{1}{2} m_{\tilde{\alpha}} \,\}\ , \\
\chi^\pm_b &= \{\, m_1\,, m_{34}\,, 0\,, m_{45}\,, m_6\,;\ \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_4) \,\}\ , \\
\chi^\pm_c &= \{\, m_{1,3}\,, m_4\,, 0\,, m_{35}\,, m_6\,;\ \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{34}) \,\}\ , \\
\chi^\pm_{c'} &= \{\, m_1\,, m_{35}\,, 0\,, m_4\,, m_{56}\,;\ \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{45}) \,\}\ , \\
\chi^\pm_d &= \{\, m_3\,, m_4\,, 0\,, m_{1,35}\,, m_6\,;\ \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{1,34}) \,\}\ , \\
\chi^\pm_{d'} &= \{\, m_1\,, m_{36}\,, 0\,, m_4\,, m_5\,;\ \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{46}) \,\}\ , \\
\chi^\pm_{\tilde{d}} &= \{\, m_{1,3}\,, m_{45}\,, 0\,, m_{34}\,, m_{56}\,;\ \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{35}) \,\}\ , \\
\chi^\pm_e &= \{\, m_3\,, m_{45}\,, 0\,, m_{1,34}\,, m_{56}\,;\ \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{1,35}) \,\}\ , \\
\chi^\pm_{e'} &= \{\, m_{1,3}\,, m_{46}\,, 0\,, m_{34}\,, m_5\,;\ \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{36}) \,\}\ , \\
\chi^\pm_{\tilde{e}} &= \{\, m_3\,, m_{46}\,, 0\,, m_{1,34}\,, m_5\,;\ \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{1,36}) \,\}\ , \\
\chi^\pm_g &= \{\, m_{34}\,, m_5\,, m_4\,, m_{1,3}\,, m_{46}\,;\ \pm\tfrac{1}{2} m_{36} \,\}\ , \\
\chi^\pm_{g'} &= \{\, m_{1,34}\,, m_{56}\,, m_4\,, m_3\,, m_{45}\,;\ \pm\tfrac{1}{2} m_{1,35} \,\}\ , \\
\chi^\pm_{\tilde{g}} &= \{\, m_{1,34}\,, m_5\,, m_4\,, m_3\,, m_{46}\,;\ \pm\tfrac{1}{2} m_{1,36} \,\}\ , \\
\chi^\pm_{g^o} &= \{\, m_4\,, m_5\,, m_{34}\,, m_1\,, m_{36}\,;\ \pm\tfrac{1}{2} m_{46} \,\}\ , \\
\chi^\pm_{g''} &= \{\, m_{1,35}\,, m_6\,, m_{45}\,, m_3\,, m_4\,;\ \pm\tfrac{1}{2} m_{1,34} \,\}\ , \\
\chi^\pm_{\hat{k}} &= \{\, m_{34}\,, m_{56}\,, m_4\,, m_{1,3}\,, m_{45}\,;\ \pm\tfrac{1}{2} m_{35} \,\}\ , \\
\chi^\pm_j &= \{\, m_4\,, m_{56}\,, m_{34}\,, m_1\,, m_{35}\,;\ \pm\tfrac{1}{2} m_{45} \,\}\ , \\
\chi^\pm_{j'} &= \{\, m_{35}\,, m_6\,, m_{45}\,, m_{1,3}\,, m_4\,;\ \pm\tfrac{1}{2} m_{34} \,\}\ , \\
\chi^\pm_{j^o} &= \{\, 0\,, m_{45}\,, m_3\,, m_1\,, m_{36,4}\,;\ \pm\tfrac{1}{2} m_{56} \,\}\ ,
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
\chi^\pm_{j''} &= \{\, m_{1,35,4}\,, m_6\,, m_5\,, m_{34}\,, 0\,;\ \pm\tfrac12 m_{1,3}\,\}\ ,\\
\chi^\pm_{k} &= \{\, 0\,, m_{46}\,, m_3\,, m_1\,, m_{35,4}\,;\ \pm\tfrac12 m_5\,\}\ ,\\
\chi^\pm_{k'} &= \{\, m_{35,4}\,, m_6\,, m_5\,, m_{1,34}\,, 0\,;\ \pm\tfrac12 m_3\,\}\ ,\\
\chi^\pm_{\tilde k} &= \{\, m_{45}\,, m_6\,, m_{35}\,, m_1\,, m_{34}\,;\ \pm\tfrac12 m_4\,\}\ ,\\
\chi^\pm_{k^o} &= \{\, 0\,, m_4\,, m_3\,, m_1\,, m_{36,45}\,;\ \pm\tfrac12 m_6\,\}\ ,\\
\chi^\pm_{k''} &= \{\, m_{1,35,34}\,, m_6\,, m_5\,, m_4\,, 0\,;\ \pm\tfrac12 m_1\,\}
\end{aligned}$$

These multiplets are depicted on Fig. 2. To save space we give only the $\Lambda^-(\chi)$ modules, since (as we know from the Main type, Fig. 1.), the $\Lambda^+(\chi)$ part of the multiplets has the same structure.

Here the ER χ^+_0 contains limits of the (anti)holomorphic discrete series representations. This is guaranteed by the fact that for this ER all Harish-Chandra parameters for non-compact roots are non-positive, i.e., $n_\alpha \leq 0$, for α from (3.4). The conformal weight has the restriction $d = \frac12(11 + m_{\tilde\alpha}) \geq 10$.

There are other limiting cases, where there are zero entries for the first five n_i values. In these cases the induction procedure would not use finite-dimensional irreps of the $su(5,1)$ subgroup. The corresponding ERs would not have direct physical meaning, however, the fact that they are together with the physically meaningful ERs has important bearing on the structure of the latter.

4.2.2. Reduced type R3 multiplets

The multiplets of reduced type R3 contain 49 ERs/GVMs and may be obtained formally from the main type by setting $m_3 = 0$. Their signatures are given explicitly by:

$$\begin{aligned}
\chi^\pm_0 &= \{\, m_1\,, 0\,, m_4\,, m_5\,, m_6\,;\ \pm\tfrac12 m_{\tilde\alpha}\,\}\ , \qquad (4.5)\\
\chi^\pm_a &= \{\, m_1\,, 0\,, m_{2,4}\,, m_5\,, m_6\,;\ \pm\tfrac12 (m_{\tilde\alpha} - m_2)\,\}\ ,\\
\chi^\pm_c &= \{\, m_1\,, m_4\,, m_2\,, m_{45}\,, m_6\,;\ \pm\tfrac12 (m_{\tilde\alpha} - m_{2,4})\,\}\ ,\\
\chi^\pm_d &= \{\, 0\,, m_4\,, m_2\,, m_{1,45}\,, m_6\,;\ \pm\tfrac12 (m_{\tilde\alpha} - m_{12,4})\,\}\ ,\\
\chi^\pm_{\tilde d} &= \{\, m_1\,, m_{45}\,, m_2\,, m_4\,, m_{56}\,;\ \pm\tfrac12 (m_{\tilde\alpha} - m_{2,45})\,\}\ ,\\
\chi^\pm_e &= \{\, 0\,, m_{45}\,, m_2\,, m_{1,4}\,, m_{56}\,;\ \pm\tfrac12 (m_{\tilde\alpha} - m_{12,45})\,\}\ ,\\
\chi^\pm_{e'} &= \{\, m_1\,, m_{46}\,, m_2\,, m_4\,, m_5\,;\ \pm\tfrac12 (m_{\tilde\alpha} - m_{2,46})\,\}\ ,\\
\chi^\pm_{\tilde e} &= \{\, 0\,, m_{46}\,, m_2\,, m_{1,4}\,, m_5\,;\ \pm\tfrac12 (m_{\tilde\alpha} - m_{12,46})\,\}\ ,\\
\chi^\pm_f &= \{\, m_4\,, m_5\,, m_{2,4}\,, m_1\,, m_{46}\,;\ \pm\tfrac12 (m_{\tilde\alpha} - m_{12,45,4})\,\}\ ,\\
\chi^\pm_{f'} &= \{\, m_{1,4}\,, m_{56}\,, m_{2,4}\,, 0\,, m_{45}\,;\ \pm\tfrac12 (m_{\tilde\alpha} - m_{2,46,4})\,\}\ ,\\
\chi^\pm_{\tilde f} &= \{\, m_{1,4}\,, m_5\,, m_{2,4}\,, 0\,, m_{46}\,;\ \pm\tfrac12 (m_{\tilde\alpha} - m_{2,45,4})\,\}\ ,\\
\chi^\pm_{f''} &= \{\, m_{1,45}\,, m_6\,, m_{2,45}\,, 0\,, m_4\,;\ \pm\tfrac12 (m_{\tilde\alpha} - m_{2,46,45})\,\}\ ,
\end{aligned}$$

$$\begin{aligned}
\chi^\pm_{g} &= \{\, m_{2,4}\,, m_5\,, m_4\,, m_1\,, m_{2,46}\,;\ \pm\tfrac{1}{2} m_{46} \,\}\ ,\\
\chi^\pm_{g'} &= \{\, m_{12,4}\,, m_{56}\,, m_4\,, 0\,, m_{2,45}\,;\ \pm\tfrac{1}{2} m_{1,45} \,\}\ ,\\
\chi^\pm_{\tilde g} &= \{\, m_{12,4}\,, m_5\,, m_4\,, 0\,, m_{2,46}\,;\ \pm\tfrac{1}{2} m_{1,46} \,\}\ ,\\
\chi^\pm_{g''} &= \{\, m_{12,45}\,, m_6\,, m_{45}\,, 0\,, m_{2,4}\,;\ \pm\tfrac{1}{2} m_{1,4} \,\}\ ,\\
\chi^\pm_{h'} &= \{\, m_{45}\,, m_6\,, m_{2,45}\,, m_1\,, m_4\,;\ \pm\tfrac{1}{2} m_{2,4} \,\}\ ,\\
\chi^\pm_{\tilde h} &= \{\, m_4\,, m_{56}\,, m_{2,4}\,, m_1\,, m_{45}\,;\ \pm\tfrac{1}{2} m_{2,45} \,\}\ ,\\
\chi^\pm_{\hat k} &= \{\, m_{2,4}\,, m_{56}\,, m_4\,, m_1\,, m_{2,45}\,;\ \pm\tfrac{1}{2} m_{45} \,\}\ ,\\
\chi^\pm_{j^o} &= \{\, m_2\,, m_{45}\,, 0\,, m_1\,, m_{2,46,4}\,;\ \pm\tfrac{1}{2} m_{56} \,\}\ ,\\
\chi^\pm_{k} &= \{\, m_2\,, m_{46}\,, 0\,, m_1\,, m_{2,45,4}\,;\ \pm\tfrac{1}{2} m_5 \,\}\ ,\\
\chi^\pm_{k'} &= \{\, m_{2,45,4}\,, m_6\,, m_5\,, m_{1,4}\,, m_2\,;\ 0 \,\}\ ,\\
\chi^\pm_{\tilde k} &= \{\, m_{2,45}\,, m_6\,, m_{45}\,, m_1\,, m_{2,4}\,;\ \pm\tfrac{1}{2} m_4 \,\}\ ,\\
\chi^\pm_{k^o} &= \{\, m_2\,, m_4\,, 0\,, m_1\,, m_{2,46,45}\,;\ \pm\tfrac{1}{2} m_6 \,\}\ ,\\
\chi^\pm_{k''} &= \{\, m_{12,45,4}\,, m_6\,, m_5\,, m_4\,, m_2\,;\ \pm\tfrac{1}{2} m_1 \,\}
\end{aligned}$$

In these multiplets the number of members is odd, since the ERs $\chi^\pm_{k'}$ coincide. These multiplets are depicted on Fig. 3.

4.2.3. Reduced type R4 multiplets

The multiplets of reduced type R4 contain 51 ERs/GVMs and may be obtained formally from the main type by setting $m_4 = 0$. Their signatures are given explicitly by:

$$\begin{aligned}
\chi^\pm_{0} &= \{\, m_1\,, m_3\,, 0\,, m_5\,, m_6\,;\ \pm\tfrac{1}{2} m_{\tilde\alpha} \,\}\ , \qquad (4.6)\\
\chi^\pm_{b} &= \{\, m_1\,, m_3\,, m_2\,, m_5\,, m_6\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_2) \,\}\ ,\\
\chi^\pm_{c} &= \{\, m_{1,3}\,, 0\,, m_2\,, m_{3,5}\,, m_6\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{23}) \,\}\ ,\\
\chi^\pm_{c'} &= \{\, m_1\,, m_{3,5}\,, m_2\,, 0\,, m_{56}\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{2,5}) \,\}\ ,\\
\chi^\pm_{d} &= \{\, m_3\,, 0\,, m_2\,, m_{1,3,5}\,, m_6\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{13}) \,\}\ ,\\
\chi^\pm_{d'} &= \{\, m_1\,, m_{3,56}\,, m_2\,, 0\,, m_5\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{2,56}) \,\}\ ,\\
\chi^\pm_{f} &= \{\, m_3\,, m_5\,, m_2\,, m_{1,3}\,, m_{56}\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{13,5}) \,\}\ ,\\
\chi^\pm_{f'} &= \{\, m_{1,3}\,, m_{56}\,, m_2\,, m_3\,, m_5\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{23,56}) \,\}\ ,\\
\chi^\pm_{\tilde f} &= \{\, m_{1,3}\,, m_5\,, m_2\,, m_3\,, m_{56}\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{23,5}) \,\}\ ,\\
\chi^\pm_{f^o} &= \{\, 0\,, m_5\,, m_{23}\,, m_1\,, m_{3,56}\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{13,5,3}) \,\}\ ,\\
\chi^\pm_{f''} &= \{\, m_{1,3,5}\,, m_6\,, m_{2,5}\,, m_3\,, 0\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{23,56,5}) \,\}\ ,\\
\chi^\pm_{g} &= \{\, m_{23}\,, m_5\,, 0\,, m_{1,3}\,, m_{2,56}\,;\ \pm\tfrac{1}{2} m_{3,56} \,\}\ ,\\
\chi^\pm_{g'} &= \{\, m_{13}\,, m_{56}\,, 0\,, m_3\,, m_{2,5}\,;\ \pm\tfrac{1}{2} m_{1,3,5} \,\}\ ,\\
\chi^\pm_{\tilde g} &= \{\, m_{13}\,, m_5\,, 0\,, m_3\,, m_{2,56}\,;\ \pm\tfrac{1}{2} m_{1,3,56} \,\}\ ,\\
\chi^\pm_{g^o} &= \{\, m_2\,, m_5\,, m_3\,, m_1\,, m_{23,56}\,;\ \pm\tfrac{1}{2} m_{56} \,\}\ ,
\end{aligned}$$

$$
\begin{aligned}
\chi^\pm_{g''} &= \{\, m_{13,5}\,, m_6\,, m_5\,, m_3\,, m_2\,;\ \pm\tfrac{1}{2}m_{1,3}\,\}\ ,\\
\chi^\pm_{h} &= \{\, 0\,, m_{56}\,, m_{23}\,, m_1\,, m_{3,5}\,;\ \pm\tfrac{1}{2}m_{2,5}\,\}\ ,\\
\chi^\pm_{h'} &= \{\, m_{3,5}\,, m_6\,, m_{2,5}\,, m_{1,3}\,, 0\,;\ \pm\tfrac{1}{2}m_{23}\,\}\ ,\\
\chi^\pm_{\tilde h} &= \{\, m_3\,, m_{56}\,, m_2\,, m_{1,3}\,, m_5\,;\ \pm\tfrac{1}{2}m_{23,5}\,\}\ ,\\
\chi^\pm_{\hat h} &= \{\, m_5\,, m_6\,, m_{23,5}\,, m_1\,, m_3\,;\ \pm\tfrac{1}{2}m_2\,\}\ ,\\
\chi^\pm_{\hat k} &= \{\, m_{23}\,, m_{56}\,, 0\,, m_{1,3}\,, m_{2,3,5}\,;\ \pm\tfrac{1}{2}m_{3,5}\,\}\ ,\\
\chi^\pm_{k} &= \{\, m_2\,, m_{56}\,, m_3\,, m_1\,, m_{23,5}\,;\ \pm\tfrac{1}{2}m_5\,\}\ ,\\
\chi^\pm_{k'} &= \{\, m_{23,5}\,, m_6\,, m_5\,, m_{1,3}\,, m_2\,;\ \pm\tfrac{1}{2}m_3\,\}\ ,\\
\chi^\pm_{\tilde k} &= \{\, m_{2,5}\,, m_6\,, m_{3,5}\,, m_1\,, m_{23}\,;\ 0\,\}\ ,\\
\chi^\pm_{k^o} &= \{\, m_2\,, 0\,, m_3\,, m_1\,, m_{23,56,5}\,;\ \pm\tfrac{1}{2}m_6\,\}\ ,\\
\chi^\pm_{k''} &= \{\, m_{13,5,3}\,, m_6\,, m_5\,, 0\,, m_2\,;\ \pm\tfrac{1}{2}m_1\,\}
\end{aligned}
$$

These multiplets are depicted on Fig. 4.

4.2.4. Reduced type R1 multiplets

The multiplets of reduced type R1 contain 49 ERs/GVMs and may be obtained formally from the main type by setting $m_1 = 0$. Their signatures are given explicitly by:

$$
\begin{aligned}
\chi^\pm_{0} &= \{\, 0\,, m_3\,, m_4\,, m_5\,, m_6\,;\ \pm\tfrac{1}{2}m_{\tilde\alpha}\,\}\ , \qquad (4.7)\\
\chi^\pm_{a} &= \{\, 0\,, m_3\,, m_{2,4}\,, m_5\,, m_6\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha}-m_2)\,\}\ ,\\
\chi^\pm_{b} &= \{\, 0\,, m_{34}\,, m_2\,, m_{45}\,, m_6\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{2,4})\,\}\ ,\\
\chi^\pm_{c'} &= \{\, 0\,, m_{35}\,, m_2\,, m_4\,, m_{56}\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{2,45})\,\}\ ,\\
\chi^\pm_{d} &= \{\, m_3\,, m_4\,, m_2\,, m_{35}\,, m_6\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{24})\,\}\ ,\\
\chi^\pm_{d'} &= \{\, 0\,, m_{36}\,, m_2\,, m_4\,, m_5\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{2,46})\,\}\ ,\\
\chi^\pm_{e} &= \{\, m_3\,, m_{45}\,, m_2\,, m_{34}\,, m_{56}\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{25})\,\}\ ,\\
\chi^\pm_{\tilde e} &= \{\, m_3\,, m_{46}\,, m_2\,, m_{34}\,, m_5\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{26})\,\}\ ,\\
\chi^\pm_{f} &= \{\, m_{34}\,, m_5\,, m_{2,4}\,, m_3\,, m_{46}\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{25,4})\,\}\ ,\\
\chi^\pm_{f^o} &= \{\, m_4\,, m_5\,, m_{24}\,, 0\,, m_{36}\,;\ \pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{25,34})\,\}\ ,\\
\chi^\pm_{g} &= \{\, m_{24}\,, m_5\,, m_4\,, m_3\,, m_{2,46}\,;\ \pm\tfrac{1}{2}m_{36}\,\}\ ,\\
\chi^\pm_{g^o} &= \{\, m_{2,4}\,, m_5\,, m_{34}\,, 0\,, m_{26}\,;\ \pm\tfrac{1}{2}m_{46}\,\}\ ,\\
\chi^\pm_{h} &= \{\, m_4\,, m_{56}\,, m_{24}\,, 0\,, m_{35}\,;\ \pm\tfrac{1}{2}m_{2,45}\,\}\ ,\\
\chi^\pm_{h'} &= \{\, m_{35}\,, m_6\,, m_{2,45}\,, m_3\,, m_4\,;\ \pm\tfrac{1}{2}m_{24}\,\}\ ,\\
\chi^\pm_{\tilde h} &= \{\, m_{34}\,, m_{56}\,, m_{2,4}\,, m_3\,, m_{45}\,;\ \pm\tfrac{1}{2}m_{25}\,\}\ ,\\
\chi^\pm_{\hat h} &= \{\, m_{45}\,, m_6\,, m_{25}\,, 0\,, m_{34}\,;\ \pm\tfrac{1}{2}m_{2,4}\,\}\ ,\\
\chi^\pm_{\hat k} &= \{\, m_{24}\,, m_{56}\,, m_4\,, m_3\,, m_{2,45}\,;\ \pm\tfrac{1}{2}m_{35}\,\}\ ,\\
\chi^\pm_{j} &= \{\, m_{2,4}\,, m_{56}\,, m_{34}\,, 0\,, m_{25}\,;\ \pm\tfrac{1}{2}m_{45}\,\}\ ,
\end{aligned}
$$

$$\begin{aligned}
\chi^\pm_{j'} &= \{ m_{25}\,, m_6\,, m_{45}\,, m_3\,, m_{2,4}\,;\, \pm\tfrac{1}{2} m_{34} \}\,, \\
\chi^\pm_{j^o} &= \{ m_2\,, m_{45}\,, m_3\,, 0\,, m_{26,4}\,;\, \pm\tfrac{1}{2} m_{56} \}\,, \\
\chi^\pm_{k} &= \{ m_2\,, m_{46}\,, m_3\,, 0\,, m_{25,4}\,;\, \pm\tfrac{1}{2} m_5 \}\,, \\
\chi^\pm_{k'} &= \{ m_{25,4}\,, m_6\,, m_5\,, m_{34}\,, m_2\,;\, \pm\tfrac{1}{2} m_3 \}\,, \\
\chi^\pm_{\tilde{k}} &= \{ m_{2,45}\,, m_6\,, m_{35}\,, 0\,, m_{24}\,;\, \pm\tfrac{1}{2} m_4 \}\,, \\
\chi^\pm_{k^o} &= \{ m_2\,, m_4\,, m_3\,, 0\,, m_{26,45}\,;\, \pm\tfrac{1}{2} m_6 \}\,, \\
\chi^\pm_{k''} &= \{ m_{25,34}\,, m_6\,, m_5\,, m_4\,, m_2\,;\, 0 \}
\end{aligned}$$

These multiplets are depicted on Fig. 5.

4.3. Doubly reduced types of multiplets

4.3.1. Type R24

Here belong multiplets of further reduction, first of reduced type R24 containing 31 ERs/GVMs which may be obtained formally from the R2 type by setting $m_4 = 0$. Their signatures are given explicitly by:

$$\begin{aligned}
\chi^\pm_{b} &= \{ m_1\,, m_3\,, 0\,, m_5\,, m_6\,;\, \pm\tfrac{1}{2} m_{\tilde{\alpha}} \}\,, \qquad (4.8)\\
\chi^\pm_{c} &= \{ m_{1,3}\,, 0\,, 0\,, m_{3,5}\,, m_6\,;\, \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_3) \}\,, \\
\chi^\pm_{c'} &= \{ m_1\,, m_{35}\,, 0\,, 0\,, m_{56}\,;\, \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_5) \}\,, \\
\chi^\pm_{d} &= \{ m_3\,, 0\,, 0\,, m_{1,3,5}\,, m_6\,;\, \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{1,3}) \}\,, \\
\chi^\pm_{d'} &= \{ m_1\,, m_{3,56}\,, 0\,, 0\,, m_5\,;\, \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{56}) \}\,, \\
\chi^\pm_{g} &= \{ m_3\,, m_5\,, 0\,, m_{1,3}\,, m_{56}\,;\, \pm\tfrac{1}{2} m_{3,56} \}\,, \\
\chi^\pm_{g'} &= \{ m_{1,3}\,, m_{56}\,, 0\,, m_3\,, m_5\,;\, \pm\tfrac{1}{2} m_{1,3,5} \}\,, \\
\chi^\pm_{\tilde{g}} &= \{ m_{1,3}\,, m_5\,, 0\,, m_3\,, m_{56}\,;\, \pm\tfrac{1}{2} m_{1,3,56} \}\,, \\
\chi^\pm_{g^o} &= \{ 0\,, m_5\,, m_3\,, m_1\,, m_{3,56}\,;\, \pm\tfrac{1}{2} m_{56} \}\,, \\
\chi^\pm_{g''} &= \{ m_{1,3,5}\,, m_6\,, m_5\,, m_3\,, 0;\, \pm\tfrac{1}{2} m_{1,3} \}\,, \\
\chi^\pm_{\hat{k}} &= \{ m_3\,, m_{56}\,, 0\,, m_{1,3}\,, m_5\,;\, \pm\tfrac{1}{2} m_{3,5} \}\,, \\
\chi^\pm_{j} &= \{ 0\,, m_{56}\,, m_3\,, m_1\,, m_{3,5}\,;\, \pm\tfrac{1}{2} m_5 \}\,, \\
\chi^\pm_{j'} &= \{ m_{3,5}\,, m_6\,, m_5\,, m_{1,3}\,, 0;\, \pm\tfrac{1}{2} m_3 \}\,, \\
\chi^\pm_{\tilde{k}} &= \{ m_5\,, m_6\,, m_{3,5}\,, m_1\,, m_3\,;\, 0 \}\,, \\
\chi^\pm_{k^o} &= \{ 0\,, 0\,, m_3\,, m_1\,, m_{3,56,5}\,;\, \pm\tfrac{1}{2} m_6 \}\,, \\
\chi^\pm_{k''} &= \{ m_{1,3,5,3}\,, m_6\,, m_5\,, 0\,, 0;\, \pm\tfrac{1}{2} m_1 \}
\end{aligned}$$

These multiplets are depicted on Fig. 6.

4.3.2. Type R23

Here belong multiplets of another reduction line, first of reduced type R23 containing 35 ERs/GVMs which may be obtained formally from the R2 type by setting $m_3 = 0$. Their signatures are given explicitly by:

$$\begin{aligned}
\chi_a^\pm &= \{\, m_1\,, 0\,, m_4\,, m_5\,, m_6\,;\, \pm\tfrac{1}{2} m_{\tilde\alpha}\,\}\ , \\
\chi_c^\pm &= \{\, m_1\,, m_4\,, 0\,, m_{45}\,, m_6\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_4)\,\}\ , \\
\chi_d^\pm &= \{\, 0\,, m_4\,, 0\,, m_{1,45}\,, m_6\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{1,4})\,\}\ , \\
\chi_{\tilde d}^\pm &= \{\, m_1\,, m_{45}\,, 0\,, m_4\,, m_{56}\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{45})\,\}\ , \\
\chi_e^\pm &= \{\, 0\,, m_{45}\,, 0\,, m_{1,4}\,, m_{56}\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{1,45})\,\}\ , \\
\chi_{e'}^\pm &= \{\, m_1\,, m_{46}\,, 0\,, m_4\,, m_5\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{46})\,\}\ , \\
\chi_{\tilde e}^\pm &= \{\, 0\,, m_{46}\,, 0\,, m_{1,4}\,, m_5\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{1,46})\,\}\ , \\
\chi_g^\pm &= \{\, m_4\,, m_5\,, m_4\,, m_1\,, m_{46}\,;\, \pm\tfrac{1}{2} m_{46}\,\}\ , \\
\chi_{g'}^\pm &= \{\, m_{1,4}\,, m_{56}\,, m_4\,, 0\,, m_{45}\,;\, \pm\tfrac{1}{2} m_{1,45}\,\}\ , \\
\chi_{\tilde g}^\pm &= \{\, m_{1,4}\,, m_5\,, m_4\,, 0\,, m_{46}\,;\, \pm\tfrac{1}{2} m_{1,46}\,\}\ , \\
\chi_{g''}^\pm &= \{\, m_{1,45}\,, m_6\,, m_{45}\,, 0\,, m_4\,;\, \pm\tfrac{1}{2} m_{1,4}\,\}\ , \\
\chi_{\hat k}^\pm &= \{\, m_4\,, m_{56}\,, m_4\,, m_1\,, m_{45}\,;\, \pm\tfrac{1}{2} m_{45}\,\}\ , \\
\chi_{j^o}^\pm &= \{\, 0\,, m_{45}\,, 0\,, m_1\,, m_{46,4}\,;\, \pm\tfrac{1}{2} m_{56}\,\}\ , \\
\chi_k^\pm &= \{\, 0\,, m_{46}\,, 0\,, m_1\,, m_{45,4}\,;\, \pm\tfrac{1}{2} m_5\,\}\ , \\
\chi_{k'}^\pm &= \{\, m_{45,4}\,, m_6\,, m_5\,, m_{1,4}\,, 0\,;\, 0\,\}\ , \\
\chi_{\tilde k}^\pm &= \{\, m_{45}\,, m_6\,, m_{45}\,, m_1\,, m_4\,;\, \pm\tfrac{1}{2} m_4\,\}\ , \\
\chi_{k^o}^\pm &= \{\, 0\,, m_4\,, 0\,, m_1\,, m_{46,45}\,;\, \pm\tfrac{1}{2} m_6\,\}\ , \\
\chi_{k''}^\pm &= \{\, m_{1,45,4}\,, m_6\,, m_5\,, m_4\,, 0\,;\, \pm\tfrac{1}{2} m_1\,\}
\end{aligned} \tag{4.9}$$

These multiplets are depicted on Fig. 7.

4.3.3. Type R21

Here belong multiplets of another reduction line, first of reduced type R21 containing 35 ERs/GVMs which may be obtained formally from the R2 type by setting $m_1 = 0$. Their signatures are given explicitly by:

$$\begin{aligned}
\chi_a^\pm &= \{\, 0\,, m_3\,, m_4\,, m_5\,, m_6\,;\, \pm\tfrac{1}{2} m_{\tilde\alpha}\,\}\ , \\
\chi_b^\pm &= \{\, 0\,, m_{34}\,, 0\,, m_{45}\,, m_6\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_4)\,\}\ , \\
\chi_c^\pm &= \{\, m_3\,, m_4\,, 0\,, m_{35}\,, m_6\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{34})\,\}\ , \\
\chi_{c'}^\pm &= \{\, 0\,, m_{35}\,, 0\,, m_4\,, m_{56}\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{45})\,\}\ , \\
\chi_{d'}^\pm &= \{\, 0\,, m_{36}\,, 0\,, m_4\,, m_5\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{46})\,\}\ , \\
\chi_{\tilde d}^\pm &= \{\, m_3\,, m_{45}\,, 0\,, m_{34}\,, m_{56}\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{35})\,\}\ , \\
\chi_{e'}^\pm &= \{\, m_3\,, m_{46}\,, 0\,, m_{34}\,, m_5\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{36})\,\}\ , \\
\chi_{g'}^\pm &= \{\, m_{34}\,, m_{56}\,, m_4\,, m_3\,, m_{45}\,;\, \pm\tfrac{1}{2} m_{35}\,\}\ , \\
\chi_{\tilde g}^\pm &= \{\, m_{34}\,, m_5\,, m_4\,, m_3\,, m_{46}\,;\, \pm\tfrac{1}{2} m_{36}\,\}\ , \\
\chi_{g^o}^\pm &= \{\, m_4\,, m_5\,, m_{34}\,, 0\,, m_{36}\,;\, \pm\tfrac{1}{2} m_{46}\,\}\ ,
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
\chi^\pm_{g''} &= \{\, m_{35}\,, m_6\,, m_{45}\,, m_3\,, m_4\,;\, \pm\tfrac{1}{2}m_{34} \,\}\,, \\
\chi^\pm_{j} &= \{\, m_4\,, m_{56}\,, m_{34}\,, 0\,, m_{35}\,;\, \pm\tfrac{1}{2}m_{45} \,\}\,, \\
\chi^\pm_{j^o} &= \{\, 0\,, m_{45}\,, m_3\,, 0\,, m_{36,4}\,;\, \pm\tfrac{1}{2}m_{56} \,\}\,, \\
\chi^\pm_{k} &= \{\, 0\,, m_{46}\,, m_3\,, 0\,, m_{35,4}\,;\, \pm\tfrac{1}{2}m_5 \,\}\,, \\
\chi^\pm_{k'} &= \{\, m_{35,4}\,, m_6\,, m_5\,, m_{34}\,, 0\,;\, \pm\tfrac{1}{2}m_3 \,\}\,, \\
\chi^\pm_{\tilde{k}} &= \{\, m_{45}\,, m_6\,, m_{35}\,, 0\,, m_{34}\,;\, \pm\tfrac{1}{2}m_4 \,\}\,, \\
\chi^\pm_{k^o} &= \{\, 0\,, m_4\,, m_3\,, 0\,, m_{36,45}\,;\, \pm\tfrac{1}{2}m_6 \,\}\,, \\
\chi^\pm_{k''} &= \{\, m_{35,34}\,, m_6\,, m_5\,, m_4\,, 0\,;\, 0 \,\}
\end{aligned}$$

These multiplets are depicted on Fig. 8.

4.3.4. Type R41

The multiplets of reduced type R41 contain 36 ERs/GVMs and may be obtained formally from the type R1 by setting $m_4 = 0$ (or from type R4 by setting $m_1 = 0$). Their signatures are given explicitly by:

$$\begin{aligned}
\chi^\pm_{0} &= \{\, 0\,, m_3\,, 0\,, m_5\,, m_6\,;\, \pm\tfrac{1}{2}m_{\tilde{\alpha}} \,\}\,, \qquad (4.11) \\
\chi^\pm_{b} &= \{\, 0\,, m_3\,, m_2\,, m_5\,, m_6\,;\, \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_2) \,\}\,, \\
\chi^\pm_{c'} &= \{\, 0\,, m_{3,5}\,, m_2\,, 0\,, m_{56}\,;\, \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{2,5}) \,\}\,, \\
\chi^\pm_{d} &= \{\, m_3\,, 0\,, m_2\,, m_{3,5}\,, m_6\,;\, \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{23}) \,\}\,, \\
\chi^\pm_{d'} &= \{\, 0\,, m_{3,56}\,, m_2\,, 0\,, m_5\,;\, \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{2,56}) \,\}\,, \\
\chi^\pm_{f} &= \{\, m_3\,, m_5\,, m_2\,, m_3\,, m_{56}\,;\, \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{23,5}) \,\}\,, \\
\chi^\pm_{f^o} &= \{\, 0\,, m_5\,, m_{23}\,, 0\,, m_{3,56}\,;\, \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{23,5,3}) \,\}\,, \\
\chi^\pm_{g} &= \{\, m_{23}\,, m_5\,, 0\,, m_3\,, m_{2,56}\,;\, \pm\tfrac{1}{2}m_{3,56} \,\}\,, \\
\chi^\pm_{g^o} &= \{\, m_2\,, m_5\,, m_3\,, 0\,, m_{23,56}\,;\, \pm\tfrac{1}{2}m_{56} \,\}\,, \\
\chi^\pm_{h} &= \{\, 0\,, m_{56}\,, m_{23}\,, 0\,, m_{3,5}\,;\, \pm\tfrac{1}{2}m_{2,5} \,\}\,, \\
\chi^\pm_{h'} &= \{\, m_{3,5}\,, m_6\,, m_{2,5}\,, m_3\,, 0\,;\, \pm\tfrac{1}{2}m_{23} \,\}\,, \\
\chi^\pm_{\tilde{h}} &= \{\, m_3\,, m_{56}\,, m_2\,, m_3\,, m_5\,;\, \pm\tfrac{1}{2}m_{23,5} \,\}\,, \\
\chi^\pm_{\hat{h}} &= \{\, m_5\,, m_6\,, m_{23,5}\,, 0\,, m_3\,;\, \pm\tfrac{1}{2}m_2 \,\}\,, \\
\chi^\pm_{\hat{k}} &= \{\, m_{23}\,, m_{56}\,, 0\,, m_3\,, m_{2,5}\,;\, \pm\tfrac{1}{2}m_{3,5} \,\}\,, \\
\chi^\pm_{k} &= \{\, m_2\,, m_{56}\,, m_3\,, 0\,, m_{23,5}\,;\, \pm\tfrac{1}{2}m_5 \,\}\,, \\
\chi^\pm_{k'} &= \{\, m_{23,5}\,, m_6\,, m_5\,, m_3\,, m_2\,;\, \pm\tfrac{1}{2}m_3 \,\}\,, \\
\chi^\pm_{\tilde{k}} &= \{\, m_{2,5}\,, m_6\,, m_{3,5}\,, 0\,, m_{23}\,;\, 0 \,\}\,, \\
\chi^\pm_{k^o} &= \{\, m_2\,, 0\,, m_3\,, 0\,, m_{23,56,5}\,;\, \pm\tfrac{1}{2}m_6 \,\}\,, \\
\chi^\pm_{k''} &= \{\, m_{23,5,3}\,, m_6\,, m_5\,, 0\,, m_2\,;\, 0 \,\}
\end{aligned}$$

These multiplets are depicted on Fig. 9.

4.3.5. Type R43

The multiplets of reduced type R43 contain 31 ERs/GVMs and may be obtained formally from the R3 type by setting $m_4 = 0$ (or from type R4 by setting $m_3 = 0$). Their signatures are given explicitly by:

$$\begin{aligned}
\chi_0^\pm &= \{\, m_1\,, 0\,, 0\,, m_5\,, m_6\,;\, \pm\tfrac{1}{2} m_{\tilde\alpha}\,\}\ , \qquad (4.12)\\
\chi_c^\pm &= \{\, m_1\,, 0\,, m_2\,, m_5\,, m_6\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_2)\,\}\ ,\\
\chi_d^\pm &= \{\, 0\,, 0\,, m_2\,, m_{1,5}\,, m_6\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{12})\,\}\ ,\\
\chi_f^\pm &= \{\, 0\,, m_5\,, m_2\,, m_1\,, m_{56}\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{12,5})\,\}\ ,\\
\chi_{f'}^\pm &= \{\, m_1\,, m_{56}\,, m_2\,, 0\,, m_5\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{2,56})\,\}\ ,\\
\chi_{\tilde f}^\pm &= \{\, m_1\,, m_5\,, m_2\,, 0\,, m_{56}\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{2,5})\,\}\ ,\\
\chi_{f''}^\pm &= \{\, m_{1,5}\,, m_6\,, m_{2,5}\,, 0\,, 0\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{2,56,5})\,\}\ ,\\
\chi_g^\pm &= \{\, m_2\,, m_5\,, 0\,, m_1\,, m_{2,56}\,;\, \pm\tfrac{1}{2} m_{56}\,\}\ ,\\
\chi_{g'}^\pm &= \{\, m_{12}\,, m_{56}\,, 0\,, 0\,, m_{2,5}\,;\, \pm\tfrac{1}{2} m_{1,5}\,\}\ ,\\
\chi_{\tilde g}^\pm &= \{\, m_{12}\,, m_5\,, 0\,, 0\,, m_{2,56}\,;\, \pm\tfrac{1}{2} m_{1,56}\,\}\ ,\\
\chi_{h'}^\pm &= \{\, m_5\,, m_6\,, m_{2,5}\,, m_1\,, 0\,;\, \pm\tfrac{1}{2} m_2\,\}\ ,\\
\chi_{\tilde h}^\pm &= \{\, 0\,, m_{56}\,, m_2\,, m_1\,, m_5\,;\, \pm\tfrac{1}{2} m_{2,5}\,\}\ ,\\
\chi_k^\pm &= \{\, m_2\,, m_{56}\,, 0\,, m_1\,, m_{2,5}\,;\, \pm\tfrac{1}{2} m_5\,\}\ ,\\
\chi_{\tilde k}^\pm &= \{\, m_{2,5}\,, m_6\,, m_5\,, m_1\,, m_2\,;\, 0\,\}\ ,\\
\chi_{k^o}^\pm &= \{\, m_2\,, 0\,, 0\,, m_1\,, m_{2,56,5}\,;\, \pm\tfrac{1}{2} m_6\,\}\ ,\\
\chi_{k''}^\pm &= \{\, m_{12,5}\,, m_6\,, m_5\,, 0\,, m_2\,;\, \pm\tfrac{1}{2} m_1\,\}
\end{aligned}$$

These multiplets are depicted on Fig. 10.

4.3.6. Type R13

The multiplets of reduced type R13 contain 29 ERs/GVMs and may be obtained formally from the R1 type by setting $m_3 = 0$. Their signatures are given explicitly by:

$$\begin{aligned}
\chi_0^\pm &= \{\, 0\,, 0\,, m_4\,, m_5\,, m_6\,;\, \pm\tfrac{1}{2} m_{\tilde\alpha}\,\}\ , \qquad (4.13)\\
\chi_a^\pm &= \{\, 0\,, 0\,, m_{2,4}\,, m_5\,, m_6\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_2)\,\}\ ,\\
\chi_d^\pm &= \{\, 0\,, m_4\,, m_2\,, m_{45}\,, m_6\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{2,4})\,\}\ ,\\
\chi_e^\pm &= \{\, 0\,, m_{45}\,, m_2\,, m_4\,, m_{56}\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{2,45})\,\}\ ,\\
\chi_{\tilde e}^\pm &= \{\, 0\,, m_{46}\,, m_2\,, m_4\,, m_5\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{2,46})\,\}\ ,\\
\chi_f^\pm &= \{\, m_4\,, m_5\,, m_{2,4}\,, 0\,, m_{46}\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{2,45,4})\,\}\ ,\\
\chi_g^\pm &= \{\, m_{2,4}\,, m_5\,, m_4\,, 0\,, m_{2,46}\,;\, \pm\tfrac{1}{2} m_{46}\,\}\ ,\\
\chi_{h'}^\pm &= \{\, m_{45}\,, m_6\,, m_{2,45}\,, 0\,, m_4\,;\, \pm\tfrac{1}{2} m_{2,4}\,\}\ ,\\
\chi_{\tilde h}^\pm &= \{\, m_4\,, m_{56}\,, m_{2,4}\,, 0\,, m_{45}\,;\, \pm\tfrac{1}{2} m_{2,45}\,\}\ ,
\end{aligned}$$

$$
\begin{aligned}
\chi^\pm_{\hat k} &= \{\, m_{2,4}\,, m_{56}\,, m_4\,, 0\,, m_{2,45}\,;\, \pm\tfrac12 m_{45} \,\}\ ,\\
\chi^\pm_{j^o} &= \{\, m_2\,, m_{45}\,, 0\,, 0\,, m_{2,46,4}\,;\, \pm\tfrac12 m_{56} \,\}\ ,\\
\chi^\pm_{k} &= \{\, m_2\,, m_{46}\,, 0\,, 0\,, m_{2,45,4}\,;\, \pm\tfrac12 m_5 \,\}\ ,\\
\chi^\pm_{k'} &= \{\, m_{2,45,4}\,, m_6\,, m_5\,, m_4\,, m_2\,;\, 0 \,\}\ ,\\
\chi^\pm_{\tilde k} &= \{\, m_{2,45}\,, m_6\,, m_{45}\,, 0\,, m_{2,4}\,;\, \pm\tfrac12 m_4 \,\}\ ,\\
\chi^\pm_{k^o} &= \{\, m_2\,, m_4\,, 0\,, 0\,, m_{2,46,45}\,;\, \pm\tfrac12 m_6 \,\}
\end{aligned}
$$

These multiplets are depicted on Fig. 11.

4.3.7. Type R15

The multiplets of reduced type R15 contain 34 ERs/GVMs and may be obtained formally from the R1 type by setting $m_5 = 0$. Their signatures are given explicitly by:

$$
\begin{aligned}
\chi^\pm_{0} &= \{\, 0\,, m_3\,, m_4\,, 0\,, m_6\,;\, \pm\tfrac12 m_{\tilde\alpha} \,\}\ , \qquad (4.14)\\
\chi^\pm_{a} &= \{\, 0\,, m_3\,, m_{2,4}\,, 0\,, m_6\,;\, \pm\tfrac12 (m_{\tilde\alpha} - m_2) \,\}\ ,\\
\chi^\pm_{c'} &= \{\, 0\,, m_{34}\,, m_2\,, m_4\,, m_6\,;\, \pm\tfrac12 (m_{\tilde\alpha} - m_{2,4}) \,\}\ ,\\
\chi^\pm_{d'} &= \{\, 0\,, m_{34,6}\,, m_2\,, m_4\,, 0\,;\, \pm\tfrac12 (m_{\tilde\alpha} - m_{2,4,6}) \,\}\ ,\\
\chi^\pm_{e} &= \{\, m_3\,, m_4\,, m_2\,, m_{34}\,, m_6\,;\, \pm\tfrac12 (m_{\tilde\alpha} - m_{24}) \,\}\ ,\\
\chi^\pm_{\tilde e} &= \{\, m_3\,, m_{4,6}\,, m_2\,, m_{34}\,, 0\,;\, \pm\tfrac12 (m_{\tilde\alpha} - m_{24,6}) \,\}\ ,\\
\chi^\pm_{f} &= \{\, m_{34}\,, 0\,, m_{2,4}\,, m_3\,, m_{4,6}\,;\, \pm\tfrac12 (m_{\tilde\alpha} - m_{24,4}) \,\}\ ,\\
\chi^\pm_{f^o} &= \{\, m_4\,, 0\,, m_{24}\,, 0\,, m_{34,6}\,;\, \pm\tfrac12 (m_{\tilde\alpha} - m_{24,34}) \,\}\ ,\\
\chi^\pm_{g} &= \{\, m_{24}\,, 0\,, m_4\,, m_3\,, m_{2,4,6}\,;\, \pm\tfrac12 m_{34,6} \,\}\ ,\\
\chi^\pm_{g^o} &= \{\, m_{2,4}\,, 0\,, m_{34}\,, 0\,, m_{24,6}\,;\, \pm\tfrac12 m_{4,6} \,\}\ ,\\
\chi^\pm_{h} &= \{\, m_4\,, m_6\,, m_{24}\,, 0\,, m_{34}\,;\, \pm\tfrac12 m_{2,4} \,\}\ ,\\
\chi^\pm_{\tilde h} &= \{\, m_{34}\,, m_6\,, m_{2,4}\,, m_3\,, m_4\,;\, \pm\tfrac12 m_{24} \,\}\ ,\\
\chi^\pm_{\hat k} &= \{\, m_{24}\,, m_6\,, m_4\,, m_3\,, m_{2,4}\,;\, \pm\tfrac12 m_{34} \,\}\ ,\\
\chi^\pm_{k} &= \{\, m_2\,, m_{4,6}\,, m_3\,, 0\,, m_{24,4}\,;\, 0 \,\}\ ,\\
\chi^\pm_{k'} &= \{\, m_{24,4}\,, m_6\,, 0\,, m_{34}\,, m_2\,;\, \pm\tfrac12 m_3 \,\}\ ,\\
\chi^\pm_{\tilde k} &= \{\, m_{2,4}\,, m_6\,, m_{34}\,, 0\,, m_{24}\,;\, \pm\tfrac12 m_4 \,\}\ ,\\
\chi^\pm_{k^o} &= \{\, m_2\,, m_4\,, m_3\,, 0\,, m_{24,6,4}\,;\, \pm\tfrac12 m_6 \,\}\ ,\\
\chi^\pm_{k''} &= \{\, m_{24,34}\,, m_6\,, 0\,, m_4\,, m_2\,;\, 0 \,\}
\end{aligned}
$$

These multiplets are depicted on Fig. 12.

4.3.8. Type R16

The multiplets of reduced type R16 contain 34 ERs/GVMs and may be obtained formally from the R1 type by setting $m_6 = 0$. Their signatures are given explicitly by:

$$\begin{aligned}
\chi_0^\pm &= \{\,0\,,m_3\,,m_4\,,m_5\,,0\,;\,\pm\tfrac{1}{2}m_{\tilde\alpha}\,\}\,, \qquad (4.15)\\
\chi_a^\pm &= \{\,0\,,m_3\,,m_{2,4}\,,m_5\,,0\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_2)\,\}\,,\\
\chi_b^\pm &= \{\,0\,,m_{34}\,,m_2\,,m_{45}\,,0\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{2,4})\,\}\,,\\
\chi_d^\pm &= \{\,m_3\,,m_4\,,m_2\,,m_{35}\,,0\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{24})\,\}\,,\\
\chi_{d'}^\pm &= \{\,0\,,m_{35}\,,m_2\,,m_4\,,m_5\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{2,45})\,\}\,,\\
\chi_{\tilde e}^\pm &= \{\,m_3\,,m_{45}\,,m_2\,,m_{34}\,,m_5\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{25})\,\}\,,\\
\chi_h^\pm &= \{\,m_4\,,m_5\,,m_{24}\,,0\,,m_{35}\,;\,\pm\tfrac{1}{2}m_{2,45}\,\}\,,\\
\chi_{h'}^\pm &= \{\,m_{35}\,,0\,,m_{2,45}\,,m_3\,,m_4\,;\,\pm\tfrac{1}{2}m_{24}\,\}\,,\\
\chi_{\tilde h}^\pm &= \{\,m_{34}\,,m_5\,,m_{2,4}\,,m_3\,,m_{45}\,;\,\pm\tfrac{1}{2}m_{25}\,\}\,,\\
\chi_{\hat h}^\pm &= \{\,m_{45}\,,0\,,m_{25}\,,0\,,m_{34}\,;\,\pm\tfrac{1}{2}m_{2,4}\,\}\,,\\
\chi_{\hat k}^\pm &= \{\,m_{24}\,,m_5\,,m_4\,,m_3\,,m_{2,45}\,;\,\pm\tfrac{1}{2}m_{35}\,\}\,,\\
\chi_j^\pm &= \{\,m_{2,4}\,,m_5\,,m_{34}\,,0\,,m_{25}\,;\,\pm\tfrac{1}{2}m_{45}\,\}\,,\\
\chi_{j'}^\pm &= \{\,m_{25}\,,0\,,m_{45}\,,m_3\,,m_{2,4}\,;\,\pm\tfrac{1}{2}m_{34}\,\}\,,\\
\chi_k^\pm &= \{\,m_2\,,m_{45}\,,m_3\,,0\,,m_{25,4}\,;\,\pm\tfrac{1}{2}m_5\,\}\,,\\
\chi_{k'}^\pm &= \{\,m_{25,4}\,,0\,,m_5\,,m_{34}\,,m_2\,;\,\pm\tfrac{1}{2}m_3\,\}\,,\\
\chi_{\tilde k}^\pm &= \{\,m_{2,45}\,,0\,,m_{35}\,,0\,,m_{24}\,;\,\pm\tfrac{1}{2}m_4\,\}\,,\\
\chi_{k^o}^\pm &= \{\,m_2\,,m_4\,,m_3\,,0\,,m_{25,45}\,;\,0\,\}\,,\\
\chi_{k''}^\pm &= \{\,m_{25,34}\,,0\,,m_5\,,m_4\,,m_2\,;\,0\,\}
\end{aligned}$$

These multiplets are depicted on Fig. 13.

4.3.9. Type R35

The multiplets of reduced type R35 contain 34 ERs/GVMs and may be obtained formally from the R3 type by setting $m_5 = 0$. Their signatures are given explicitly by:

$$\begin{aligned}
\chi_0^\pm &= \{\,m_1\,,0\,,m_4\,,0\,,m_6\,;\,\pm\tfrac{1}{2}m_{\tilde\alpha}\,\}\,, \qquad (4.16)\\
\chi_a^\pm &= \{\,m_1\,,0\,,m_{2,4}\,,0\,,m_6\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_2)\,\}\,,\\
\chi_{\tilde d}^\pm &= \{\,m_1\,,m_4\,,m_2\,,m_4\,,m_6\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{2,4})\,\}\,,\\
\chi_e^\pm &= \{\,0\,,m_4\,,m_2\,,m_{1,4}\,,m_6\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{12,4})\,\}\,,\\
\chi_{e'}^\pm &= \{\,m_1\,,m_{4,6}\,,m_2\,,m_4\,,0\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{2,4,6})\,\}\,,\\
\chi_{\tilde e}^\pm &= \{\,0\,,m_{4,6}\,,m_2\,,m_{1,4}\,,0\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{12,4,6})\,\}\,,\\
\chi_f^\pm &= \{\,m_4\,,0\,,m_{2,4}\,,m_1\,,m_{4,6}\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{12,4,4})\,\}\,,\\
\chi_{f'}^\pm &= \{\,m_{1,4}\,,m_6\,,m_{2,4}\,,0\,,m_4\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{2,4,6,4})\,\}\,,\\
\chi_{\tilde f}^\pm &= \{\,m_{1,4}\,,0\,,m_{2,4}\,,0\,,m_{4,6}\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{2,4,4})\,\}\,,\\
\chi_g^\pm &= \{\,m_{2,4}\,,0\,,m_4\,,m_1\,,m_{2,4,6}\,;\,\pm\tfrac{1}{2}m_{4,6}\,\}\,,
\end{aligned}$$

$$\begin{aligned}
\chi^\pm_{g'} &= \{\, m_{12,4}\,, m_6\,, m_4\,, 0\,, m_{2,4}\,;\, \pm\tfrac{1}{2} m_{1,4}\,\}\ ,\\
\chi^\pm_{\tilde g} &= \{\, m_{12,4}\,, 0\,, m_4\,, 0\,, m_{2,4,6}\,;\, \pm\tfrac{1}{2} m_{1,4,6}\,\}\ ,\\
\chi^\pm_{\tilde h} &= \{\, m_4\,, m_6\,, m_{2,4}\,, m_1\,, m_{45}\,;\, \pm\tfrac{1}{2} m_{2,4}\,\}\ ,\\
\chi^\pm_{k} &= \{\, m_2\,, m_{4,6}\,, 0\,, m_1\,, m_{2,4,4}\,;\, 0\,\}\ ,\\
\chi^\pm_{k'} &= \{\, m_{2,4,4}\,, m_6\,, 0\,, m_{1,4}\,, m_2\,;\, 0\,\}\ ,\\
\chi^\pm_{\tilde k} &= \{\, m_{2,4}\,, m_6\,, m_4\,, m_1\,, m_{2,4}\,;\, \pm\tfrac{1}{2} m_4\,\}\ ,\\
\chi^\pm_{k^o} &= \{\, m_2\,, m_4\,, 0\,, m_1\,, m_{2,4,6,4}\,;\, \pm\tfrac{1}{2} m_6\,\}\ ,\\
\chi^\pm_{k''} &= \{\, m_{12,4,4}\,, m_6\,, 0\,, m_4\,, m_2\,;\, \pm\tfrac{1}{2} m_1\,\}
\end{aligned}$$

These multiplets are depicted on Fig. 14.

4.4. Triply reduced type of multiplets

Here we give only multiplets that contain at least one non-limit ER, since the others would not be relevant for the applications.

4.4.1. Type R235

Next comes reduced type R235 containing 24 ERs/GVMs which may be obtained formally from the R23 type by setting $m_5 = 0$. Their signatures are given explicitly by:

$$\begin{aligned}
\chi^\pm_{a} &= \{\, m_1\,, 0\,, m_4\,, 0\,, m_6\,;\, \pm\tfrac{1}{2} m_{\tilde\alpha}\,\}\ ,\\
\chi^\pm_{\tilde d} &= \{\, m_1\,, m_4\,, 0\,, m_4\,, m_6\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_4)\,\}\ ,\\
\chi^\pm_{e} &= \{\, 0\,, m_4\,, 0\,, m_{1,4}\,, m_6\,;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{1,4})\,\}\ ,\\
\chi^\pm_{e'} &= \{\, m_1\,, m_{4,6}\,, 0\,, m_4\,, 0;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{4,6})\,\}\ ,\\
\chi^\pm_{\tilde e} &= \{\, 0\,, m_{4,6}\,, 0\,, m_{1,4}\,, 0;\, \pm\tfrac{1}{2}(m_{\tilde\alpha} - m_{1,4,6})\,\}\ ,\\
\chi^\pm_{g} &= \{\, m_4\,, 0\,, m_4\,, m_1\,, m_{4,6}\,;\, \pm\tfrac{1}{2} m_{4,6}\,\}\ ,\\
\chi^\pm_{g'} &= \{\, m_{1,4}\,, m_6\,, m_4\,, 0\,, m_4\,;\, \pm\tfrac{1}{2} m_{1,4}\,\}\ ,\\
\chi^\pm_{\tilde g} &= \{\, m_{1,4}\,, 0\,, m_4\,, 0\,, m_{4,6}\,;\, \pm\tfrac{1}{2} m_{1,4,6}\,\}\ ,\\
\chi^\pm_{k} &= \{\, 0\,, m_{4,6}\,, 0\,, m_1\,, m_{4,4}\,;\, 0\,\}\ ,\\
\chi^\pm_{k'} &= \{\, m_{4,4}\,, m_6\,, 0\,, m_{1,4}\,, 0;\, 0\,\}\ ,\\
\chi^\pm_{\tilde k} &= \{\, m_4\,, m_6\,, m_4\,, m_1\,, m_4\,;\, \pm\tfrac{1}{2} m_4\,\}\ ,\\
\chi^\pm_{k^o} &= \{\, 0\,, m_4\,, 0\,, m_1\,, m_{4,4,6}\,;\, \pm\tfrac{1}{2} m_6\,\}\ ,\\
\chi^\pm_{k''} &= \{\, m_{1,4,4}\,, m_6\,, 0\,, m_4\,, 0;\, \pm\tfrac{1}{2} m_1\,\}
\end{aligned} \tag{4.17}$$

These multiplets are depicted on Fig. 15.

4.4.2. Type R215

Next comes reduced type R215 containing 24 ERs/GVMs which may be obtained formally from the R21 type by setting $m_5 = 0$. Their signatures are given explicitly by:

$$\begin{aligned}
\chi_a^\pm &= \{\,0\,,m_3\,,m_4\,,0\,,m_6\,;\,\pm\tfrac{1}{2}m_{\tilde\alpha}\,\}\,,\\
\chi_{c'}^\pm &= \{\,0\,,m_{34}\,,0\,,m_4\,,m_6\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_4)\,\}\,,\\
\chi_{d'}^\pm &= \{\,0\,,m_{34,6}\,,0\,,m_4\,,0\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{4,6})\,\}\,,\\
\chi_{\tilde d}^\pm &= \{\,m_3\,,m_4\,,0\,,m_{34}\,,m_6\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{34})\,\}\,,\\
\chi_{e'}^\pm &= \{\,m_3\,,m_{4,6}\,,0\,,m_{34}\,,0\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{34,6})\,\}\,,\\
\chi_{g'}^\pm &= \{\,m_{34}\,,m_6\,,m_4\,,m_3\,,m_4\,;\,\pm\tfrac{1}{2}m_{34}\,\}\,,\\
\chi_{\tilde g}^\pm &= \{\,m_{34}\,,0\,,m_4\,,m_3\,,m_{4,6}\,;\,\pm\tfrac{1}{2}m_{34,6}\,\}\,,\\
\chi_{g^o}^\pm &= \{\,m_4\,,0\,,m_{34}\,,0\,,m_{34,6}\,;\,\pm\tfrac{1}{2}m_{4,6}\,\}\,,\\
\chi_k^\pm &= \{\,0\,,m_{4,6}\,,m_3\,,0\,,m_{34,4}\,;\,0\,\}\,,\\
\chi_{k'}^\pm &= \{\,m_{34,4}\,,m_6\,,0\,,m_{34}\,,0\,;\,\pm\tfrac{1}{2}m_3\,\}\,,\\
\chi_{\tilde k}^\pm &= \{\,m_4\,,m_6\,,m_{34}\,,0\,,m_{34}\,;\,\pm\tfrac{1}{2}m_4\,\}\,,\\
\chi_{k^o}^\pm &= \{\,0\,,m_4\,,m_3\,,0\,,m_{34,6,4}\,;\,\pm\tfrac{1}{2}m_6\,\}\,,\\
\chi_{k''}^\pm &= \{\,m_{34,34}\,,m_6\,,0\,,m_4\,,0\,;\,0\,\}
\end{aligned} \tag{4.18}$$

These multiplets are depicted on Fig. 16.

4.4.3. Type R216

Next comes reduced type R216 containing 26 ERs/GVMs which may be obtained formally from the R21 type by setting $m_6 = 0$. Their signatures are given explicitly by:

$$\begin{aligned}
\chi_a^\pm &= \{\,0\,,m_3\,,m_4\,,m_5\,,0\,;\,\pm\tfrac{1}{2}m_{\tilde\alpha}\,\}\,,\\
\chi_b^\pm &= \{\,0\,,m_{34}\,,0\,,m_{45}\,,0\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_4)\,\}\,,\\
\chi_c^\pm &= \{\,m_3\,,m_4\,,0\,,m_{35}\,,0\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{34})\,\}\,,\\
\chi_{c'}^\pm &= \{\,0\,,m_{35}\,,0\,,m_4\,,m_5\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{45})\,\}\,,\\
\chi_{d'}^\pm &= \{\,0\,,m_{35}\,,0\,,m_4\,,m_5\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{45})\,\}\,,\\
\chi_{e'}^\pm &= \{\,m_3\,,m_{45}\,,0\,,m_{34}\,,m_5\,;\,\pm\tfrac{1}{2}(m_{\tilde\alpha}-m_{35})\,\}\,,\\
\chi_{g'}^\pm &= \{\,m_{34}\,,m_5\,,m_4\,,m_3\,,m_{45}\,;\,\pm\tfrac{1}{2}m_{35}\,\}\,,\\
\chi_{g''}^\pm &= \{\,m_{35}\,,0\,,m_{45}\,,m_3\,,m_4\,;\,\pm\tfrac{1}{2}m_{34}\,\}\,,\\
\chi_j^\pm &= \{\,m_4\,,m_5\,,m_{34}\,,0\,,m_{35}\,;\,\pm\tfrac{1}{2}m_{45}\,\}\,,\\
\chi_k^\pm &= \{\,0\,,m_{45}\,,m_3\,,0\,,m_{35,4}\,;\,\pm\tfrac{1}{2}m_5\,\}\,,\\
\chi_{k'}^\pm &= \{\,m_{35,4}\,,0\,,m_5\,,m_{34}\,,0\,;\,\pm\tfrac{1}{2}m_3\,\}\,,\\
\chi_{\tilde k}^\pm &= \{\,m_{45}\,,0\,,m_{35}\,,0\,,m_{34}\,;\,\pm\tfrac{1}{2}m_4\,\}\,,
\end{aligned} \tag{4.19}$$

$$
\begin{aligned}
\chi_{k^o}^{\pm} &= \{\, 0\,, m_4\,, m_3\,, 0\,, m_{35,45}\,;\, 0 \,\}\ , \\
\chi_{k''}^{\pm} &= \{\, m_{35,34}\,, 0\,, m_5\,, m_4\,, 0\,;\, 0 \,\}
\end{aligned}
$$

These multiplets are depicted on Fig. 17.

4.4.4. Type R416

The multiplets of reduced type R416 contain 25 ERs/GVMs and may be obtained formally from the type R41 by setting $m_6 = 0$. Their signatures are given explicitly by:

$$
\begin{aligned}
\chi_0^{\pm} &= \{\, 0\,, m_3\,, 0\,, m_5\,, 0\,;\, \pm\tfrac{1}{2} m_{\tilde{\alpha}} \,\}\ , \\
\chi_b^{\pm} &= \{\, 0\,, m_3\,, m_2\,, m_5\,, 0\,;\, \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_2) \,\}\ , \\
\chi_{c'}^{\pm} &= \{\, 0\,, m_{3,5}\,, m_2\,, 0\,, m_5\,;\, \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{2,5}) \,\}\ , \\
\chi_d^{\pm} &= \{\, m_3\,, 0\,, m_2\,, m_{3,5}\,, 0\,;\, \pm\tfrac{1}{2}(m_{\tilde{\alpha}} - m_{23}) \,\}\ , \\
\chi_f^{\pm} &= \{\, m_3\,, m_5\,, m_2\,, m_3\,, m_5\,;\, \pm\tfrac{1}{2} m_{23,5} \,\}\ , \\
\chi_{f^o}^{\pm} &= \{\, 0\,, m_5\,, m_{23}\,, 0\,, m_{3,5}\,;\, \pm\tfrac{1}{2} m_{2,5} \,\}\ , \\
\chi_g^{\pm} &= \{\, m_{23}\,, m_5\,, 0\,, m_3\,, m_{2,5}\,;\, \pm\tfrac{1}{2} m_{3,5} \,\}\ , \\
\chi_{h'}^{\pm} &= \{\, m_{3,5}\,, 0\,, m_{2,5}\,, m_3\,, 0\,;\, \pm\tfrac{1}{2} m_{23} \,\}\ , \\
\chi_{\hat{h}}^{\pm} &= \{\, m_5\,, 0\,, m_{23,5}\,, 0\,, m_3\,;\, \pm\tfrac{1}{2} m_2 \,\}\ , \\
\chi_k^{\pm} &= \{\, m_2\,, m_5\,, m_3\,, 0\,, m_{23,5}\,;\, \pm\tfrac{1}{2} m_5 \,\}\ , \\
\chi_{k'}^{\pm} &= \{\, m_{23,5}\,, 0\,, m_5\,, m_3\,, m_2\,;\, \pm\tfrac{1}{2} m_3 \,\}\ , \\
\chi_{\tilde{k}}^{\pm} &= \{\, m_{2,5}\,, 0\,, m_{3,5}\,, 0\,, m_{23}\,;\, 0 \,\}\ , \\
\chi_{k^o}^{\pm} &= \{\, m_2\,, 0\,, m_3\,, 0\,, m_{23,5,5}\,;\, 0 \,\}\ , \\
\chi_{k''}^{\pm} &= \{\, m_{23,5,3}\,, 0\,, m_5\,, 0\,, m_2\,;\, 0 \,\}
\end{aligned}
\tag{4.20}
$$

These multiplets are depicted on Fig. 18.

5. Outlook

In the present paper we continued the programme outlined in [3] on the example of the non-compact group $E_{6(-14)}$. Similar explicit descriptions are planned for the other non-compact groups, in particular those with highest/lowest weight representations. We plan also to extend these considerations to the supersymmetric cases and also to the quantum group setting. Such considerations are expected to be very useful for applications to string theory and integrable models, cf., e.g., [38].

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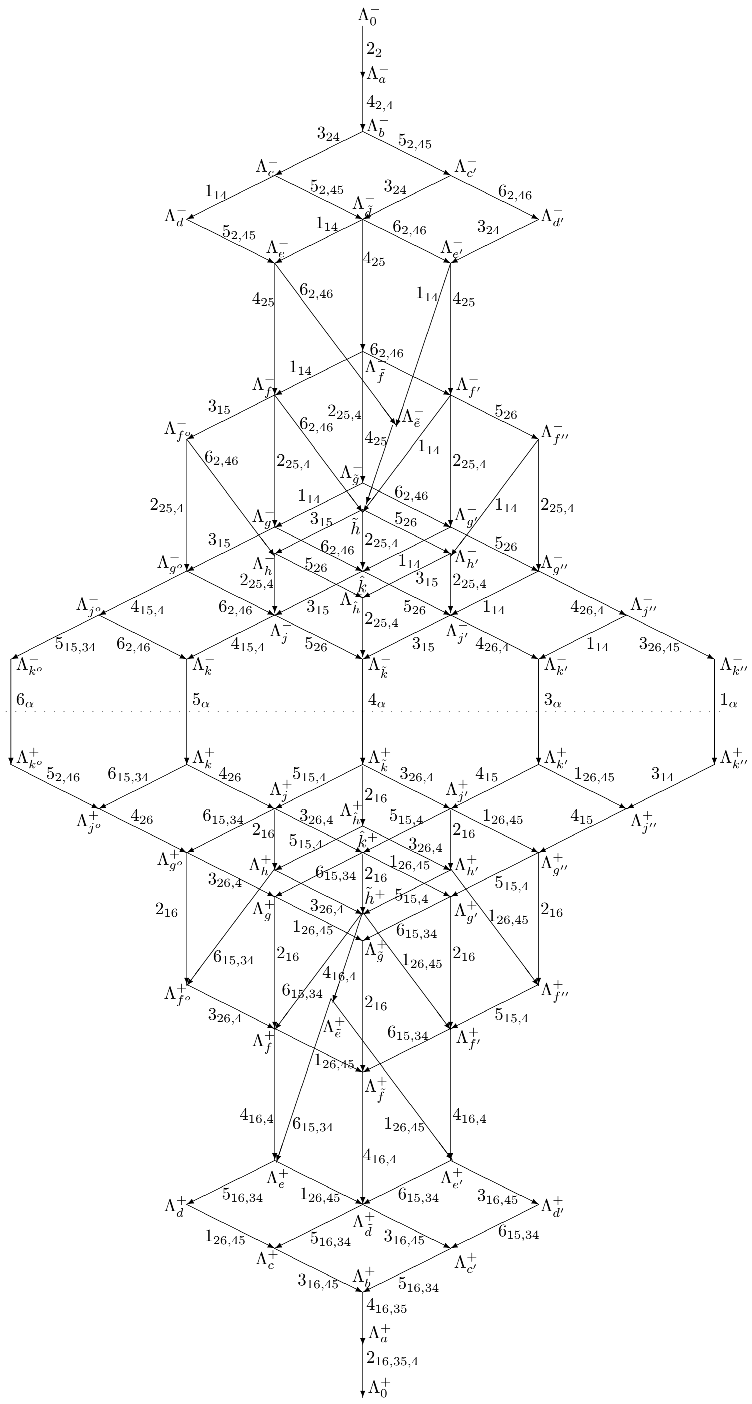


Fig. 1. Main Type

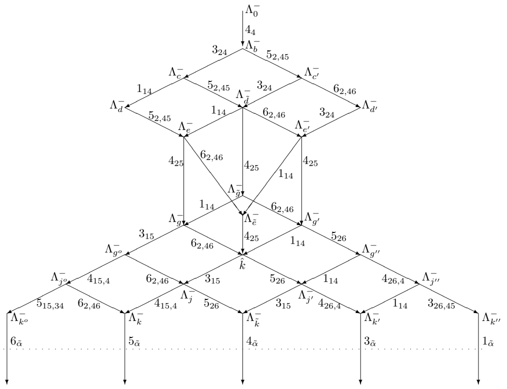


Fig. 2. Type *R2*

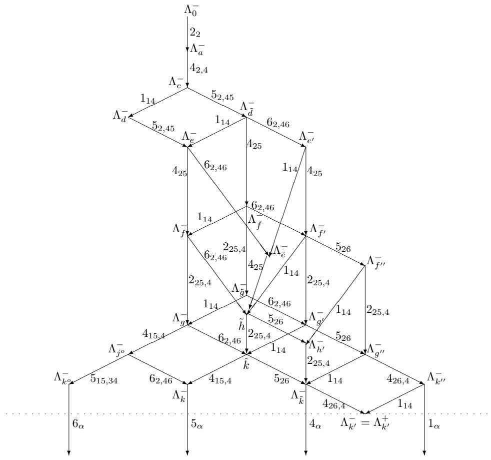


Fig. 3. Type *R3*

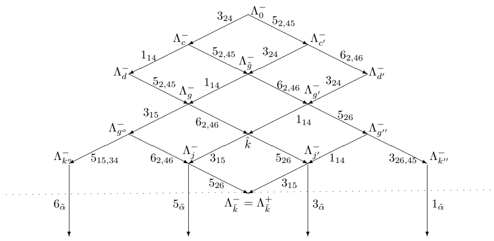


Fig. 6. Type *R24*

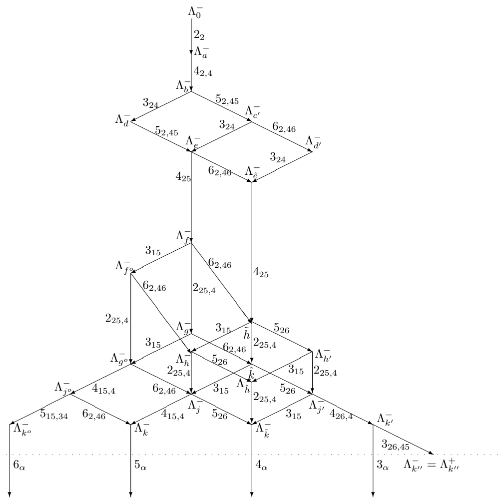


Fig. 5. Type $R1$

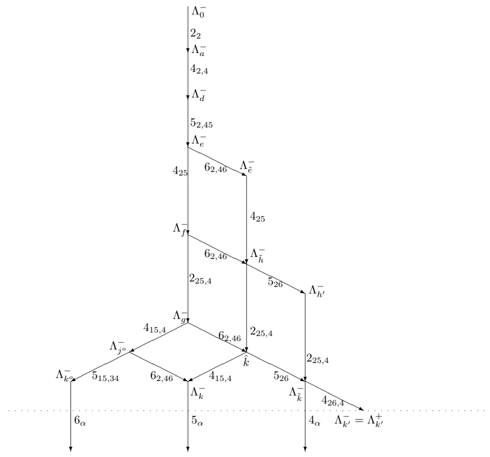


Fig. 11. Type $R13$

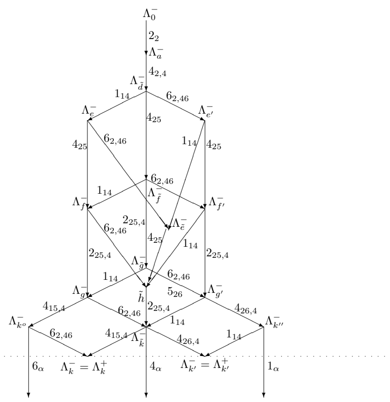


Fig. 14. Type $R35$

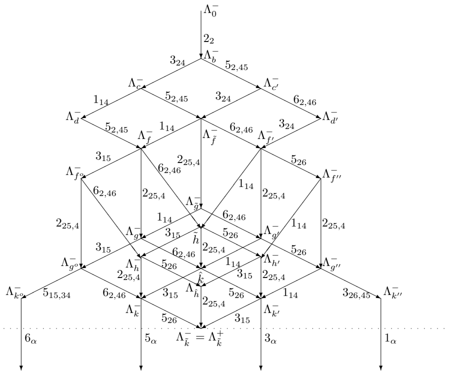


Fig. 4. Type $R4$

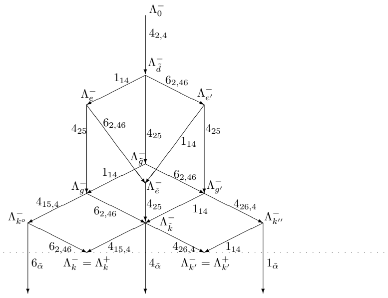


Fig. 15. Type $R235$

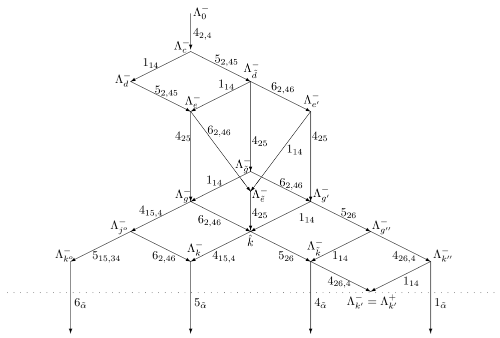


Fig. 7. Type $R23$

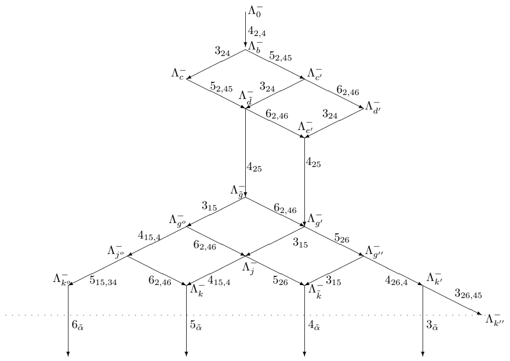


Fig. 8. Type $R21$

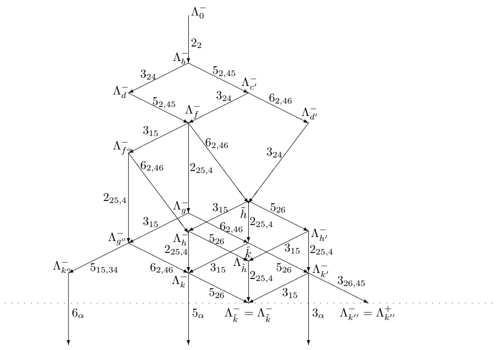
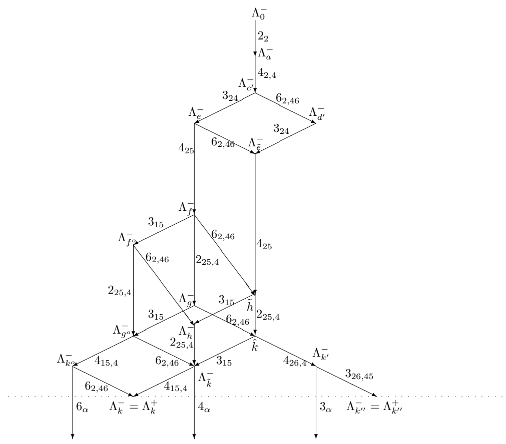


Fig. 9. Type $R41$



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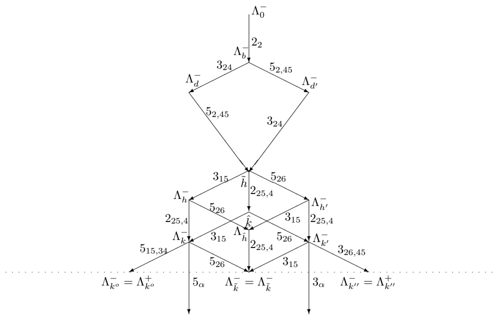

Fig. 18. Type $R416$

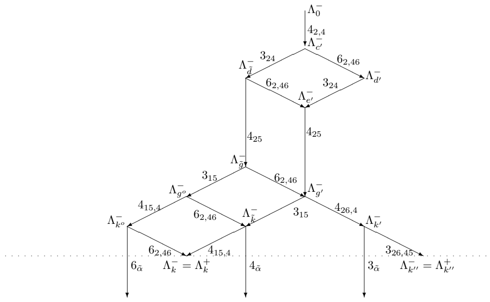

Fig. 16. Type $R215$

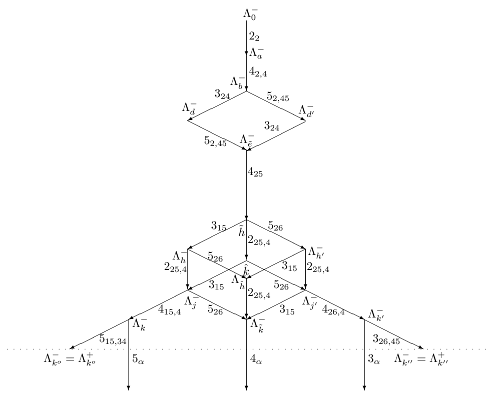

Fig. 13. Type $R16$

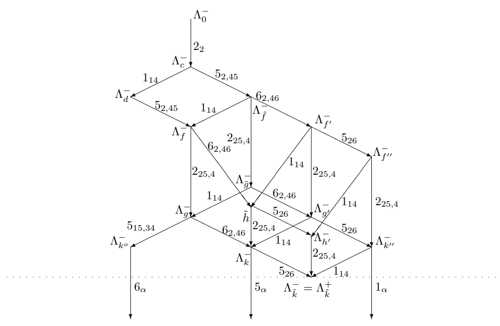

Fig. 10. Type $R43$

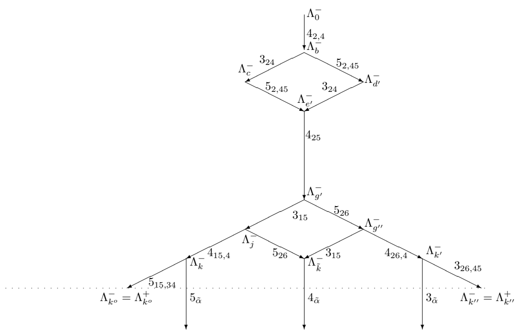

Fig. 17. Type $R216$