Twistor String Structure of the Kerr-Schild Geometry and Consistency of the Dirac-Kerr System*

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Abstract

Kerr-Schild (KS) geometry of the rotating black-holes and spinning particles is based on the associated twistor structure which is defined by the Kerr theorem and determined by analytic generating function $F(Z)$ of the projective twistor coordinates $Z \in CP^3$.

On the other hand, there is a Newman initiated complex representation which describes the source of Kerr-Newman solution as a "particle" propagating along a complex world-line $X(t+i\sigma)$, and generating function of the Kerr theorem may also be determined via parameters of this world-line, $F(Z) = F_X(Z)$. Since a complex world-line $X(t+i\sigma)$ is really a 2D worldsheet, the source of Kerr-Newman solution may be represented as a complex Euclidean string extended in the imaginary time direction σ. Due to relation $F(Z) = F_X(Z)$, the twistor KS structure turns out to be adjoined to the Kerr's complex string source, forming a natural twistor-string construction, similar to the Nair-Witten twistor-string. We show that twistor polarization of a massive Kerr-Newman particle may be matched with special solutions of the *massless* Dirac equation, providing consistency of the Dirac-Kerr model of spinning particle and forming a natural twistor-string structure concordant with the Nair-Witten concept on the scattering amplitudes in twistor space.

Introduction. The Kerr-Schild (KS) geometry is a background of the rotating black-hole (BH) solutions. On the other hand, the charged Kerr-Newman solution has $g = 2$ as that of the Dirac electron, and the KS geometry acquires central role in the problem of matching of electron with gravity [1]. KS geometry has the reach twistor structure which turns out

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to be related with the Nair-Witten concept on the scattering amplitudes in twistor space [2, 3, 4].

KS solutions in 4D are algebraically special solutions of type D which has two (doubled) Principal Null Congruences (PNC) corresponding to geodesic lines of outgoing or ingoing photons. Tangent vectors to these congruences, $k_\mu(x)^\pm$, are null and determine two different coordinate system of the black-hole solutions related to the 'in' or 'out' congruence, [5]. KS metrics have very simple KS form

$$g_{\mu\nu} = \eta_{\mu\nu} - 2Hk_\mu k_\nu, \tag{1}$$

where $\eta_{\mu\nu}$ is metric of auxiliary Minkowski space-time, and the in (or out) null vector field $k^\mu(x), \ x \in M^4$ determines symmetry of space-time, in particular, direction of gravitational 'dragging'. For rotating BH geometry the congruence is twisting which causes the difficulties for its derivation and analysis.

The obtained exact solutions of the Einstein-Maxwell system of equations [6] indicate that electromagnetic (em) field is not free with respect to the choice of the in or out representation. The direction of congruence related with em field must be the same as it is for metric, i.e. vector potential of em field A_μ has to satisfy the alignment condition, $A_\mu k^\mu = 0$. Since the classical em-fields are determined by the retarded potentials, the in-out symmetry of gravity turns out to be broken and the solutions of the Einstein-Maxwell system have to be based on the out-congruences, which has important physical consequences. The structure of Kerr congruence for the stationary Kerr-Newman solution at rest is shown in Fig.1.

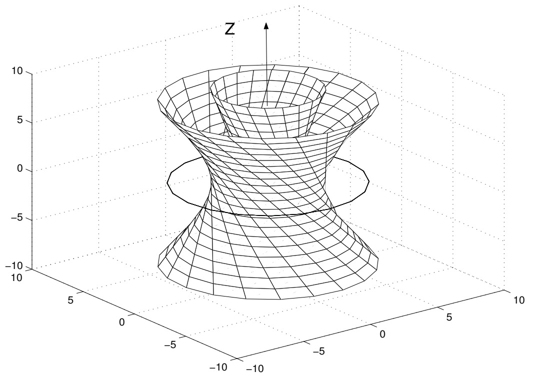

Figure 1: The Kerr singular ring and the Kerr congruence.

It displays very specific twosheetedness of the KS space-time. The Kerr singular ring represents a branch line of space-time, showing that congruence is propagating from negative sheet of metric onto positive one, and the outgoing congruence is analytic extension of the ingoing one.

The KS metrics have a rigid connection with metric of auxiliary Minkowski space-time $\eta_{\mu\nu}$ and are practically unfastened from the position of horizon. It allows one to obtain solutions which are form-invariant with respect to position of the horizons and to consider the black-holes and spinning particles in gravity (without horizon) on the common basis [7], as well as analyze deformations of the black-hole horizon [8, 9].

The Kerr theorem. Kerr congruence forms a fiber bundle of KS space-time which is determined by the Kerr theorem [10, 11, 12, 13]. The lightlike fibers of the Kerr congruence are *real* twistors (intersections of the complex conjugated null planes [11]). The Kerr theorem gives a rule to generate geodesic and shear-free (GSF) null congruences in Minkowski space-time M^4. Due to the specifical form of the KS metrics, these congruences turn out to be also geodesic and shear-free in the curved KS space-times which justifies application of the KS formalism [6] to solutions of the Einstein-Maxwell field equations.

The Kerr theorem states that any GSF congruence in M^4 is determined by some holomorphic generating function $F(Z)$ of the projective twistor coordinates

$$Z = (Y, \quad \zeta - Yv, \quad u + Y\bar{\zeta}\ ), \tag{2}$$

being an analytic solution of the equation $F(Z) = 0$ in projective twistor space, $Z \in CP^3$. [1] The variable Y plays especial role, being projective spinor coordinate $Y = \pi^2/\pi^1$ and simultaneously the projective angular coordinate

$$Y = e^{i\phi} \tan \frac{\theta}{2}. \tag{3}$$

The dependence $Y(x)$ is the output of the Kerr theorem which determines the Kerr congruence as a field of null directions

$$k_\mu dx^\mu = P^{-1}(du + \bar{Y} d\zeta + Y d\bar{\zeta} - Y\bar{Y} dv). \tag{4}$$

The null field k^μ may also be expressed in spinor form

$$k^\mu = \bar{\pi}\sigma^\mu \pi \tag{5}$$

via the Pauli matrices σ^μ, or in the terms of spinor components [12, 3, 2]

$$k_{a\dot{a}} = \sigma^\mu_{a\dot{a}} k_\mu = \pi_a \bar{\pi}_{\dot{a}}. \tag{6}$$

Each real null ray represents a twistor which is fixed by projective twistor coordinates (2), or by homogenous coordinates

$$Z^\alpha = \{\pi^a, \mu_{\dot{a}}.\} \tag{7}$$

[1] Twistor coordinates are defined via the null Cartesian coordinates $2^{\frac{1}{2}}\zeta = x + iy, \quad 2^{\frac{1}{2}}\bar{\zeta} = x - iy, \quad 2^{\frac{1}{2}}u = z - t, \quad 2^{\frac{1}{2}}v = z + t.$

The spinor π^a determines the null direction k^μ, and spinor $\mu_{\dot a} = x_\nu \sigma^\nu_{\dot a a} \pi^a$ fixes the position (equation) of a real null ray (or of the corresponding complex null plane).

Quadratic in Y functions $F(Y)$ determine two roots $Y^\pm(x)$ of the equation $F = 0$ which corresponds to twosheetedness of the Kerr space-time.

It was shown in seminal work [6] that any quadratic generating function $F(Y)$ of the Kerr theorem determines a class of the exact solutions of the Einstein-Maxwell field equations. The case of quadratic in Y functions $F(Y) = A(x)Y^2 + B(x)Y + C(x)$ was studied in details [11, 14]. In fact, it describes the Kerr-Newman solution in a general position with arbitrary spin-orientation and Lorentz boost. This information is encoded in the coefficients A, B, C, and the KS formalism allows one to write down the corresponding form of congruence and the exact solutions for metric and em field.

Complex string as source of Kerr geometry. Structure of the Kerr-Newman solution admits a complex interpretation (suggested by Newman [15]), as being generated by a source propagating along a complex world-line $X^\mu(\tau) \in CM^4$. Parameters of the quadratic in Y function $F(Y)$ may easily be determined from parameters of the world-line [11, 14]. Since a complex world-line $X^\mu(\tau)$ in CM^4 is parameterized by complex time $\tau = t + i\sigma$, it is really a complex world-sheet with target space CM^4, [16]. Thereby, the complex Kerr-Newman source is equivalent to some complex string. The KS twistor structure may be described by a complex retarded-time construction, in which twistors of the Kerr congruence are the real sections of the complex light cones emanated from the complex world-line[2]. In this way the Kerr-Newman solution may be generalized to solutions of a broken $N = 2$ supergravity [17]. Projection of the *real* KS twistors onto complex world line selects on the world-sheet a strip $Im\ \tau \in [-a, a]$, forming an open Euclidean string extended along the complex time direction $\sigma = Im \tau$. The parameter σ is linked with angular directions of twistor lines, $\sigma = a \cos\theta$.[3] Condition of the existence of real slice for the complex twistors determines the end points of the string $\tau = t \pm ia$ [16]. In the same time, consistency of boundary conditions demands orientifolding this string, by doubling of the world-sheet and turning it into a closed but folded string [16, 4]. The joined to the end points of this string twistors form two especial null rays emanated in the North and South directions, $k_N = \bar\pi_N \sigma^\mu \pi_N$ and $k_S = \bar\pi_S \sigma^\mu \pi_S$. [1]. They play peculiar role in the KS geometry, controlling parameters of the function F, and therefore, the twistorial structure in

[2] Newman suggested this construction for CM^4, [15], while the application to curved space-times demands the KS representation [11, 4].

[3] In fact, each point of the complex string has adjoined complex twistor parameter Y with the additional freedom of the rotations $e^{i\phi}$ around z-axis. Up to this $U(1)$ symmetry, each point of the complex Kerr string has an adjoined real twistor line of the Kerr congruence, forming the Kerr twistor-string structure [4].

whole. The corresponding end points of complex string may be marked by quark indices $\Psi^{\alpha}_{N(S)}$. It was obtained in [1] that the null directions k_N and k_S may be set in the one-to-one correspondence with solutions of the Dirac equation $\Psi = (\phi_\alpha, \chi^{\dot\alpha})$. Putting $\pi_N = \phi$ and $\pi_S = \bar\chi$, one sees that the Dirac wave function Ψ manages the position, orientation and boost of the Kerr source. It gives a combined Dirac-Kerr model of electron, in which twistor structure is controlled by the solutions of the Dirac equation [1]. Plane waves of the Dirac wave function propagate along the null directions k_N and k_S. The essential drawback of this model is that the Dirac equation is considered in M^4, instead of the treatment on the KS background.

Consistent solutions. Exact solutions of the massive Dirac equation on the Kerr background are unknown, and moreover, there are evidences that they cannot be obtained in principle. Note, that plane waves are also inconsistent with the Kerr background and there is a related problem on consistency of the conventional Fourier transform with the twosheeted Kerr background. In the same time, solutions of the Dirac equation may easily be obtained for the *massless* fields which are aligned with the Kerr congruence. Such solutions may be extracted from the super-Kerr-Newman solution to broken N=2 supergravity [17]. For the aligned to PNC solutions, the Dirac spinor $\Psi^\dagger_D = (\phi, \chi) = (A,\ B,\ C,\ D)$ has to satisfy the relation $\Psi_D k_\mu \gamma^\mu = 0$, which yields, [17, 18], $A = D = 0$, and the functions B and C take the form ([17], App. B) $B = f_B(\bar Y, \bar\tau)/\bar{\tilde r}, \qquad C = f_C(Y, \tau)/\tilde r, \quad \tilde r = r + ia\cos\theta$, where f_B and f_C are arbitrary analytic functions of the complex angular variable Y and the retarded-time $\tau = t - \tilde r$, obeying the relations $\tau_{,2} = \tau_{,4} = 0$, and $Y_{,2} = Y_{,4} = 0$. Due to this analyticity, the wave solutions have poles at some values of Y, or a series of such poles $f_B = \sum_i a_i/(Y - Y_i)$, leading to singular beams along some of twistor lines [4]. It turns out that the treatment of the *massless* solutions resolves four problems:

1) existence of the exact self-consistent solutions,
2) the Dirac plane waves on the KS background turn into the lightlike beams concentrating near singular twistor lines with directions k_N and k_S.
3) there appears a natural massless string structure [19] and relation to the massless Nair-Witten twistor models for scattering, [3, 2].
4) the Dirac massless solutions generate lightlike momentum $p(Y)$ distributed over sphere, $Y \in S^2$. The total nonzero mass appears by averaging over sphere[5].

[4] It is valid for em-field too, [8]

[5] The real source of Kerr-Newman solution is a disk (D2-brane). The total mass has a few contributions [7], and after integration over rays, it depends on $Y \in S^2$ which may be factorized into angular part $e^{i\phi}$, and parameter $\sigma = a\cos\theta = a\frac{1-Y\bar Y}{1+Y\bar Y}$ along the complex Kerr string.

Orientifolding is expected to be equivalent to normal ordering

$$p^\mu =<: \bar{\Psi}\gamma^\mu\Psi :>= \int_{S^2} \mu(Y) dY d\bar{Y} : \bar{\Psi}\gamma^\mu\Psi : . \tag{8}$$

In the rest frame $m =< p^0(Y) > |_{S^2}$. Therefore, mechanism of the origin of mass from the massless solutions is similar to that of the dual (super)string models and corresponds to an initial Wheeler's idea of a geon [20] ("mass without mass" obtained from the averaged em field + gravity), as well as to the initial Ramond idea on averaging of the local string structure in the dual string models [21]. The corresponding wave solutions on the Kerr background have singular beams which take asymptotically the form of the exact singular pp-wave solutions by A. Peres [22]. Note that singular null-strings (lightlike beams) along the Kerr congruence appear instead of the conventional smooth harmonic functions in several problems related with the exact solutions of the massless equation on the KS background [17, 24, 8].[6] Singular beams appear also in the multi-particle KS solutions which are described by functions $F(Y)$ of higher degrees in Y, [23]. The corresponding KS congruences form multisheeted Riemann surfaces and lead to solutions with multisheeted space-time, which generalizes the known twosheetedness of the Kerr geometry. It contradicts to classical gravity, but has far-reaching consequences for quantum gravity, enforcing us to consider metric as a quantum operator $\hat{g}_{\mu\nu}$ of an "effective geometry" $< out|\hat{g}_{\mu\nu}|in >= g_{\mu\nu} < out|in >$, [25].

The KS twistor structure and the Nair WZW twistor model. Nair considers S^2 as a momentum space of the massless gauge bosons and constructs a WZW model over sphere S^2 which is parametrized by the spinor coordinates π^a, [2]. It is similar to our field of null directions $k^\mu(\pi^a)$ in M^4. The Nair-Witten treatment of the scattering amplitudes in twistor space CP^3 means that the time evolution occurs in M^4 along twistor null lines $Z = const.$, while S^2 assumes topological duty. Twistors form a trivial bundle over S^2, $\pi : M^4 \to S^2$. Introducing two-dimensional spinor fields $\Psi(\pi)$ on the sphere, Nair considers correlation function $< \Psi_r(\pi)\Psi_s(\pi') >= \frac{\delta_{rs}}{\pi\pi'} \sim \frac{\delta_{rs}}{Y-Y'}$ and the local currents $J^a = \bar{\Psi}t^a\Psi$, showing that OPE takes in homogenous spinor coordinates the usual form of the corresponding Kac-Moody algebra of the WZW model[26] with central extension $k = 1$. In this case the complex-analytic structure of WZW model is lifted from S^2 to twistor space CP^3 over M^4. Twistor string structure of the KS geometry has much in common with the Nair WZW model and Witten's twistor string B-model, as well as many specifical features. However, in all the cases twistor approach displays the principal advantage, providing a natural extension of CFT from S^2 to M^4. Since the massless KS twistor space allows

[6] In particular, the beam-like solutions appear by electromagnetic excitations of black-holes, breaking topological structure of horizon [24, 8].

one to describe massive spinning particles, the Nair-Witten treatment of the scattering amplitudes of the gauge fields may be extended in the KS geometry to describe scattering of the gauge fields on the massive black-holes and spinning particles.

Conclusion. Consistency of the Dirac-Kerr model is to be based on the solutions of the *massless* Dirac equation aligned with KS background, which provides natural twistor string structure. These solutions were extracted from fermionic part of the super-Kerr-Newman solutions to broken $N = 2$ supergravity [17], which shows that super-KS solutions of this type reflect twistorial structure of spinning particles and may shed the light on the twistorial aspects of quantum theory matched with gravity.

References

[1] A. Burinskii, *Grav. Cosmol.* **14**(2008) 109, arXiv: hep-th/0507109.
A. Burinskii, *J. Phys A: Math. Theor,* **41**(2008) 164069, arXiv: 0710.4249[hep-th]

[2] V.P. Nair, *Phys.Lett.* **B 214** (1988) 215.

[3] E. Witten, *Comm. Math. Phys.* **252**, (2004) 189, hep-th/0312171.

[4] A. Burinskii *Phys. Rev.* **D 70**, (2004) 086006.

[5] Ch. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, W.H. Freeman and Company, San Francisco, 1973.

[6] G. C. Debney, R. P. Kerr and A. Schild, *J. Math. Phys.* **10**, (1969) 1842.

[7] A. Burinskii, *J. Phys. A: Math. Gen.* **39**(2006) 6209.
A. Burinskii, E. Elizalde, S. R. Hildebrandt and G. Magli, *Phys.Rev.* **D 65** 064039 (2002), gr-qc/0109085.

[8] A. Burinskii, E. Elizalde, S.R. Hildebrandt and G. Magli, *Phys.Rev.* **D 74**, (2006) 021502(R); *Grav. Cosmol.* **12**, (2006) 119.

[9] A. Burinskii, " Aligned Electromagnetic Excitations of the Kerr-Schild Solutions", *Proc. of the MG11 Meeting*, arXiv: gr-qc/0612186;

[10] A. Burinskii and R.P. Kerr, *Nonstationary Kerr Congruences*, gr-qc/9501012.

[11] A. Burinskii, *Phys.Rev.***D 67**, (2003) 124024; *Clas. Quant. Gravity*, **20**, (2003) 905.

[12] R. Penrose, *J. Math. Phys.* **8** (1967) 345.

[13] D.Kramer, H.Stephani, E. Herlt, M.MacCallum, "Exact Solutions of Einstein's Field Equations", Cambridge Univ. Press, 1980.

[14] A. Burinskii and G. Magli, *Phys.Rev.* **D 61** (2000) 044017.

[15] E.T. Newman, *J. Math. Phys.* **14**, (1973) 102; R.W. Lind and E.T.Newman, *J. Math. Phys.* **15**, 1103 (1974).

[16] A.Ya. Burinskii, *Phys.Lett.* **A 185** (1994) 441; *String-like Structures in Complex Kerr Geometry.* In: *Relativity Today,* Edited by R.P.Kerr and Z.Perjés, Akadémiai Kiadó, Budapest, 1994, p.149.

[17] A. Burinskii, *Phys.Rev.* **D 70**, (1998) 2392, *Class. Quant. Grav.* **16**(1999)3497.

[18] S. Einstein and R. Finkelstein, *J. Math. Phys.* **18**, (1977) 664.

[19] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, v.1, Cambridge University Press, 1987.

[20] A.Ya. Burinskii, *Sov. Phys. JETP,* **39** (1974) 193; *Russ. Phys. J.* **17** (1974) 1068.

[21] P. Ramond, *Phys.Rev.* **D 3** (1971) 2415.

[22] A. Burinskii, *Grav. Cosmol.* **10**, (2004) 50;

[23] A. Burinskii, *Grav. Cosmol.* **11**, (2005) 301; **12**, (2006) 119, *Int. J. Geom. Methods Mod. Phys.* **4**, (2007) 437.

[24] A. Burinskii, E. Elizalde, S.R. Hildebrandt and G. Magli, *Aligned electromagnetic excitations of a black hole and their impact on its quantum horizon*, arXiv:0705.3551, [hep-th].

[25] C.R. Stephens, G. t' Hooft and B.F. Whiting, *Class. Quant. Grav.* **11**(1994) 621.

[26] V.G. Knizhnik and A.B. Zamolodchikov, *Nucl.Phys.* **B247**, 83 (1984).