Counting plane Mumford curves*

Patrick Erik Bradley†
Universität Karlsruhe (TH), Englerstr. 7, 76131 Karlsruhe, GERMANY

Abstract

A p-adic version of Gromov-Witten invariants for counting plane curves of genus g and degree d through a given number of points is discussed. The multiloop version of p-adic string theory considered by Chekhov and others motivates us to ask how many of these curves are Mumford curves, i.e. uniformisable by a domain at the boundary of the Bruhat-Tits tree for $\mathrm{PGL}_2(\mathbb{Q}_p)$. Generally, the number of Mumford curves depends on the position of the given points in $\mathbb{P}^2$. With the help of tropical geometry we find configurations of points through which all curves of given degree and genus are Mumford curves. The article is preceded by an introduction to some concepts of p-adic geometry and their relation to string theory. [1]

1. Introduction

Since the work of Volovich [18] string theory has profited from p-adic methods. However, each p-adic field K has its own string theory. The consideration of classical string theory as a limit of p-adic string theories for "$p \to 1$" requires a unified approach for all p-adic number fields for fixed prime number p.

We propose p-adic geometry as a framework for realising this task. In this article, we introduce methods from this framework with string theoretic relevance. Some of these have been applied to the analysis of hierarchical data [2]. More methods are developed in p-adic enumerative geometry [3]. Of particular interest are the Mumford curves which play a role in the p-adic multiloop calculations in [6]. Conjecturally, these special curves are the only ones contributing to the string amplitude [6, Conj. 4.3.3]. From the point of view of so-called *tropical geometry*, this CMZ-conjecture, as we call it, should come natural. The reason is that tropical curves are generically obtained from transforming Mumford curves into combinatorial objects. In any case, our work is motivated by the conjecture.

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† e-mail address: bradley@math.uni-karlsruhe.de
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The aim of our methodological overview is twofold. Primarily, we want to show how they can be used to count plane Mumford curves. Secondly, we indicate how the methods could give a positive answer to a more precise formulation of the CMZ-conjecture. Our long-term goal is to be able to "predict" enumerative results for Mumford curves with p-adic string theory, similarly as in the classical case—the only difference being that the mathematical answers might be known before their physical derivations.

We refer to the article by Dragovich [8] for an introduction to p-adic numbers and their relation to string theory.

2. Prélude: An introduction to p-adic geometry

Let $\mathbb{Q}_p$ be the field of p-adic numbers. In the following, we will often use the notation $|x|$ for $|x|_K$, where K is any finite extension field of $\mathbb{Q}_p$ containing x. This notation is well defined. In fact, we could as well consider x as an element of $\mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$, and $|\ | = |\ |_{\mathbb{C}_p}$, the unique extension of $|\ |_p$ to $\mathbb{C}_p$. By O_K, we denote the ring of integers

$$O_K = \{x \in K \mid |x|_K \leq 1\},$$

and $\kappa = O_K/\pi O_K$ is the residue field. It is finite and does not depend on the choice of the *uniformiser* π which generates the maximal ideal of O_K.

The field K has an affine geometry. Hence, we can write $K = \mathbb{A}^1(K)$. However, this space is only the set of K-rational points of the geometric object $\mathbb{A}^1$ which we call *affine line*. We will often make the distinction between a space X and its K-rational points $X(K)$.

The topology of p-adic spaces such as $\mathbb{A}^1$ is totally disconnected. This uncomfortable fact can be remedied e.g. by introducing extra points. Here, we do this with the method from [1] and call the extra points *Berkovich points*. In the example of the affine line $\mathbb{A}^1$, the important Berkovich points correspond to the discs $B_a = \{|x - a| \leq r\}$ with $r > 0$.

2.1. Projective spaces

The idea of projective space is to have a good compactification of affine space which is locally affine. *Projective n-space* over K is

$$\mathbb{P}^n(K) := \left\{\text{lines through } 0 \in K^{n+1}\right\}$$

One has a decomposition $\mathbb{P}^n(K) = \mathbb{A}^n \cup \mathbb{P}^{n-1}(K)$, i.e. another projective space "at infinity". Projective coordinates are often written as

$$(x_0 : \cdots : x_n)$$

with $(x_0 : \cdots : x_n) = (y_0 : \cdots : y_n)$ if and only if there is some $\lambda \neq 0$ such that $x_i = \lambda y_i$ for all i. The local structure is given by

$$\mathbb{P}^n = U_0 \cup \cdots \cup U_n$$

with affine pieces

$$U_i = \left\{ \left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i} \right) \mid x_i \neq 0 \right\} \cong \mathbb{A}^n.$$

For example, $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ is the projective line. The projective plane is $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$. It has the property that any two lines in $\mathbb{P}^2$ intersect.

The space $\mathbb{P}^n$ is endowed in a natural way with a line bundle. Namely, for $x \in \mathbb{P}^n(K)$ let ℓ_x be the line in K^{n+1} represented by the point x. This line bundle is the *tautological line bundle* $O(1)$ encountered later on.

2.2. Bruhat-Tits tree

The symmetry group of the projective line $\mathbb{P}^1$ over K is $\mathrm{PGL}_2(K)$, the group of fractional transformations

$$z \mapsto \frac{az+b}{cz+d} \tag{1}$$

with $ad - bc \neq 0$. The map (1) is also called *Möbius transformation.* The fact that Möbius transformations take discs to discs allows to construct an infinite tree $\mathscr{T}_K$ on which $\mathrm{PGL}_2(K)$ acts as group of symmetries. This tree is the *Bruhat-Tits tree* for $\mathrm{PGL}_2(K)$ and can be visualised as the hierarchical tree of discs

$$B_a = \{x \mid |x-a|_K \leq |r|_K\},$$

the vertices being given by B_a and an edge is given by maximal strict inclusion $B_b \subset B_a$ of discs[2], i.e. any B_c such that $B_b \subseteq B_c \subseteq B_a$ satisfies either $B_c = B_b$ or $B_c = B_a$. The tree $\mathscr{T}_K$ is a $q+1$-regular tree, meaning that from each vertex there are precisely $q+1$ edges going out, where q is the cardinality of the residue field κ. The geometric reason behind this fact is that every vertex v of $\mathscr{T}_K$ corresponds to a projective line $\mathbb{P}^1_v$, and its attached edges correspond to the κ-rational points $\mathbb{P}^1_v(\kappa)$. Hence, $\mathscr{T}_K$ can be seen as representing the combinatorics of infinitely many projective lines glued together as (locally) depicted in Figure 1.

An important fact is that the boundary of the tree $\mathscr{T}_K$ is given by $\mathbb{P}^1(K)$. Namely, an infinite path in $\mathscr{T}_K$ can be understood as a strictly descending sequence of discs whose limit is their intersection: a K-rational point in $\mathbb{P}^1$.

Let now $C = \mathbb{P}^1 \setminus \{p_1, \ldots, p_n\}$ be the projective line over K with n points (also called *punctures*) $p_1, \ldots, p_n$ removed. These points define a subtree $T = \mathscr{T}(p_1, \ldots, p_n)$ of $\mathscr{T}_K$ by connecting all geodesic paths inside the Bruhat-Tits tree between the punctures. In [2], this tree was interpreted as dendrogram for the "data" $p_1, \ldots, p_n$.

[2] However, there is some subtlety concerning the invariance under $\mathrm{PGL}_2(K)$, wherefore the vertices are given by equivalence classes of discs cf. [2, §3].

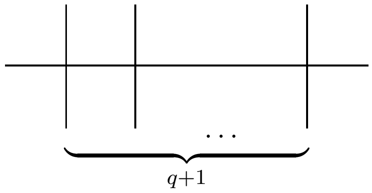

Figure 1: Tree of projective lines.

The tree T corresponds to the gluing of projective lines $\mathbb{P}^1_v$ over κ for each vertex v and then removing n punctures. This geometric object is a singular curve C_s, the singularities being ordinary double points, and the lines X_i constituting the curve C_s are the *irreducible components*. They are represented in $\mathbb{P}^1$ by discs corresponding to Berkovich points ξ_i.

2.3. Mumford curves

The p-adic analogon of Riemann surface in the physics literature is the *Mumford curve*. It allows a Schottky uniformisation: if F_g is a discrete subgroup of $\mathrm{PGL}_2(K)$ which is generated by g hyperbolic transformations, then $X = \Omega/F_g$ is a complete algebraic curve. Here, $\Omega \subseteq \mathbb{P}^1$ is the domain of regularity of the action of F_g.

Not every p-adic algebraic curve allows a Schottky uniformisation. However, there are some characterizations of Mumford curves. Namely, every p-adic curve X has a so-called *O_K-model*. It is a curve $\mathcal{X}$ defined over the ring O_K of integers of K: consider some (local) set of equations for X, and clearing all denominators yields equations with coefficients in O_K. Then reducing all equations modulo π yields a curve X_s defined over κ, called the *special fibre* of $\mathcal{X}$. In general, X_s is singular, even if X is not. By a theorem of Deligne and Mumford [7, Cor. 2.7], it is possible for K sufficiently large to find an O_K-model $\mathcal{X}$ such that X_s is a so-called *stable* curve, meaning:

- All singularities of X_s are ordinary double points.
- $|\mathrm{Aut}\, X_s| < \infty$.

There is a reduction map

$$\rho\colon X \to X_s \tag{2}$$

which is locally "reduction modulo π". The upper curve X is called the *generic fibre* of $\mathcal{X}$.

We now assume that K is sufficiently large. The characterising criterion for X being a Mumford curve is then that the special fibre X_s is a union of genus zero curves [10, Thm. 5.4.1, 5.5.5].

The special fibre X_s of a stable curve allows a combinatorial description by taking as vertices the irreducible components of X_s and as edges the double points. The resulting graph Γ is the *dual graph* of X_s. This yields the next characterisation: X is a Mumford curve, if and only if the first Betti number of the dual graph Γ of X_s equals the genus of X.

Let now X be a curve with n punctures. Then the dual graph of X can be obtained from the dual graph Γ' of the completion of X by adhering some infinitely long spines to Γ'. The result is a so-called *n-pointed tropical curve* $\Gamma = \mathrm{trop}(X)$. We call the combinatorial object underlying Γ a *semigraph*, and the spines are the *punctures*. There is a metric on Γ coming from the reduction map (2). Namely, the fibre $\rho^{-1}(x)$ of a point $x \in X_s$ is either an open disc or an open annulus A. The latter holds true, if and only if x is a double point. The length of an edge of Γ is defined as the thickness of A.

Our central mathematical object will be the *moduli space of n-pointed genus g curves* $\mathcal{M}_{g,n}$ whose points are equivalence classes $[C, p_1, \dots, p_n]$, where C is a complete curve of genus g minus n punctures $p_1, \dots, p_n$. The moduli space is defined over the integers $\mathbb{Z}$. Therefore, we advocate the use of $\mathcal{M}_{g,n}$ in adelic physics, although our focus will be on $M_{g,n} = \mathcal{M}_{g,n} \otimes K$ for K a sufficiently large extension of $\mathbb{Q}_p$.

The methods here yield a *tropicalisation map*

$$\mathrm{trop}\colon M_{g,n} \to M_{g,n}^{\mathrm{trop}}, [C, p_1, \dots, p_n] \to [\mathrm{trop}(C), p_1, \dots, p_n],$$

where $M_{g,n}^{\mathrm{trop}}$ is the moduli space of n-pointed tropical curves of genus $\leq g$. The punctures of $\mathrm{trop}(C)$ are labeled in the same way as the punctures of C. The moduli spaces are not compact. The *Deligne-Mumford compactification* $\bar{\mathcal{M}}_{g,n}$ is defined by including the stable curves. This allows to define $\bar{M}_{g,n}^{\mathrm{trop}}$ as the space parametrising tropical curves whose edge lengths can take any value between 0 and ∞. The latter comes from a singularity in the generic fibre C. In the former case, it can happen that loops get contracted to a vertex. Then $\mathrm{trop}(C)$ is not the tropicalisation of a Mumford curve, as the Betti number is lower than the genus of C (cf. also [3]).

If $g = 0$, then $\mathrm{trop}(C)$ coincides with $\mathscr{T}(p_1, \dots, p_n)$ from the previous subsection. Also the singular curve C_s considered there is the special fibre of an O_K-model $\mathcal{C}$ of C.

2.4. Tropical geometry of the p-adic projective plane

We give here a very brief introduction into the aspects of tropical geometry which we later use. A more general introduction to tropical geometry can be found e.g. in [16, 17].

The valuation map

$$\mathrm{Val}\colon (K \setminus \{0\})^2 \to \mathbb{R}^2,\ (x, y) \mapsto (v_K(x), v_K(y)),$$

with $v_K(z) = -\log|z|_K$, has as its image the lattice $\frac{1}{e}\mathbb{Z}^2$ in the Euclidean plane, where e is the ramification index of K over $\mathbb{Q}_p$. Making the p-adic field K larger results in a refinement of the lattice. In the limit, or if $K = \mathbb{C}_p$, we obtain the rational points of the Euclidean plane.

The valuation map extends to the projective plane:

$$\mathrm{Val}\colon \mathbb{P}^2 \to \mathbb{TP}^2 \tag{3}$$

by defining $v_K(0) = \infty$ on each affine patch U, i.e. we get the extra points $(\infty, y), (x, \infty), (\infty, \infty)$ on the closure of $\mathrm{Val}(U)$. The tropical projective plane $\mathbb{TP}^2$ is by definition the gluing of these closed sets. The result is homeomorphic to the 2-simplex whose interior corresponds to $\mathbb{R}^2$, and whose boundary segments correspond to the parts with a coordinate ∞. The simplex structure reflects the fact that the complement of $(K \setminus \{0\})^2$ in $\mathbb{P}^2(K)$ is the union $\mathcal{H}$ of three lines not intersecting in a common point.

However, the valuation map brings more changes. It transforms p-adic geometry to so-called *tropical geometry*, in which the objects are piecewise affine-linear spaces. For example, curves in $(K \setminus \{0\})^2$ transform to sets whose closures are tropical curves embedded in the plane [9, Thm. 2.1.1]. A tropical line in the plane is depicted in Figure 2. The three unbounded

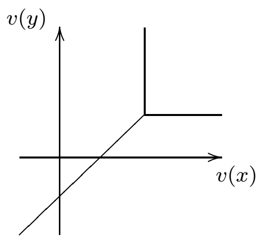

Figure 2: A tropical line in the plane.

edges are explained by the fact that any line in $\mathbb{P}^2(K)$ intersects $\mathcal{H}$ in three points. More generally, any plane curve C of degree d intersects $\mathcal{H}$ in $3d$ points. This means that the closure of $\mathrm{Val}(C)$ in $\mathbb{R}^2$ is a tropical curve with $3d$ ends (counted with multiplicity).

One successful application of tropical geometry was in providing elementary proofs to classical enumeration problems of algebraic geometry. E.g. the Kontsevich formula [14, Claim 5.2.1] for counting rational curves of degree d through $3d-1$ points in the plane was obtained by counting plane tropical curves of genus zero [11].

2.5. p-adic vs. tropical integration over p-adic spaces

Here, we want to relate two ways of integrating over p-adic spaces. The first one using the Haar measure on locally compact fields will be called *p-adic integration*. The other method, which we call *tropical integration*, is by taking a limit of measures coming from p-adic line bundles. This allows to compute integrals via tropicalisation.

p-adic integration. On the locally compact additive group $\mathbb{Q}_p$, there is a translation invariant measure dx called *Haar measure*. It is usually normalised such that

$$\int_{\mathbb{Z}_p} dx = 1.$$

This measure can be extended to a measure on $\mathbb{P}^1(\mathbb{Q}_p)$ by using the substitution

$$\phi\colon x \mapsto \frac{1}{px}, \tag{4}$$

which changes dx to $p\frac{dx}{|x|_p^2}$. We also denote it as dx and obtain

$$\int\limits_{\mathbb{P}^1(\mathbb{Q}_p)} dx = \int\limits_{\mathbb{Z}_p} dx + \int\limits_{\{x\in\mathbb{Q}_p \mid |x|>1\}} dx = 1 + p\int\limits_{\mathbb{Z}_p} \frac{dx}{|x|_p^2} = 1 + p\cdot\frac{1}{p} = 2.$$

The same holds true with any finite extension field K of $\mathbb{Q}_p$, as long as the Haar measure dx is normalised to

$$\int_{O_K} dx = 1,$$

where $O_K = \{x \in K \mid |x|_K \leq 1\}$ is the ring of integers of K.

However, if we want to allow K to vary arbitrarily among the finite extension fields of $\mathbb{Q}_p$, then it is often convenient to consider $K = \mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$. This approach gives some meaning to the limiting process "$p \to 1$" as explained in [12], where it is viewed as taking a sequence of uniformisers π_K for each K. These have the property

$$\lim_K |\pi_K|_K = \lim_{e\to\infty} p^{-\frac{1}{e}} = 1,$$

where e is the ramification index of K over $\mathbb{Q}_p$. In any case, one arrives at trying to integrate over a field which is no longer locally compact.

Tropical integration. In order to be able to integrate over a p-adic space X, as opposed to its set of K-rational points $X(K)$, we use the method of Chambert-Loir [5] for p-adic line bundles.

To the tautological line bundle $O(1)$ on p-adic $\mathbb{P}^1$ can be associated a curvature form $c_1(\bar{O}(1))$ as follows[3]: Let $\mathcal{X}$ be the union of two copies of projective lines over κ intersecting in one point as in Figure 3. It can be realised as a reduction modulo π of $\mathbb{P}^1$, and its dual graph has this shape:

[3]The notation $\bar{O}(1)$ or $\bar{L}$ stands for metrised line bundle. But we suppress the definition of the metric on L for the curvature form here.

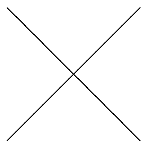

Figure 3: A possible reduction of $\mathbb{P}^1$.

•——•. This corresponds to the substitution (4), and the pre-image of the double point (represented by the open line segment) under the reduction map $\rho\colon \mathbb{P}^1 \to \mathcal{X}$ is the interior of the overlap $\mathbb{D} \cap \phi(\mathbb{D})$, where $\mathbb{D}$ is the p-adic unit disc. Let now $\mathcal{L}$ be the line bundle over $\mathcal{X}$ which is the tautological line bundle over each component. It can be seen as a reduction of $O(1) \otimes O(1) = O(1)^2$ on $\mathbb{P}^1$. Then, using the algebraic version of curvature form on X, let

$$c_1(\bar{O}(1)) := \frac{1}{2}\left(c_1(\mathcal{L}|_{X_1})\delta_{\xi_1} + c_1(\mathcal{L}|_{X_2})\delta_{\xi_2}\right),$$

where δ_{ξ_i} is the Dirac measure supported on the Berkovich point ξ_i corresponding to the component X_i. This defines a Borel measure on $\mathbb{P}^1$ which induces via ρ the measure which distributes the weight $\frac{1}{2}$ onto each endpoint of the unit interval. A careful application of a smoothing process developed by Gubler [13] yields a measure μ on $\mathbb{P}^1$ which via the tropicalisation map

$$\mathbb{P}^1 \to \mathbb{TP}^1, \quad x \mapsto -\log|x|_K,$$

induces the Lebesgue measure $d\lambda$ restricted to $\mathbb{TP}^1$ satisfying

$$\int\limits_{\mathbb{P}^1} \mu = \int\limits_{\mathbb{TP}^1} d\lambda = 1$$

[3]. This measure μ differs on K from p-adic dx only by a factor 2. However, μ has the advantage that it is well-defined over $\mathbb{C}_p$. Hence, we arrive at a tropical interpretation of the limit "$p \to 1$". We will also call μ the *tropical limit* of dx.

Let now X be a p-adic manifold of dimension d. The generalization of the method above needs d line bundles $L_1, \dots, L_d$ on X, and one obtains a regular Borel measure by the formula

$$\mu = c_1(\bar{L}_1) \wedge \cdots \wedge c_1(\bar{L}_d) = \sum_Y c_1(\mathcal{L}_1|_Y) \wedge \cdots \wedge c_1(\mathcal{L}_d|_Y)\delta_{\xi_Y}, \tag{5}$$

where Y runs through the irreducible components of the special fibre of a given O_K-model of X, and $\mathcal{L}_i$ are specialisations of O_K-models of L_i.

3. The p-adic tree-level amplitudes

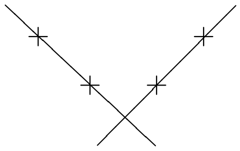

Figure 4: A stable 4-pointed genus 0 curve.

This section serves as a physical motivation for counting plane Mumford curves. The methods from the previous section are applied to the string amplitude at the tree level, and will lead in the following section to an interpretation of a conjecture by Chekhov et al. [6, Conj. 4.3.3] using more precise terms. The named authors admittedly remained vague in formulating their conjecture. Let us recall from [8, 18] the p-adic 4-point Veneziano string amplitude

$$A_p^0(k_1, k_2, k_3, k_4) = \int_{\mathbb{Q}_p} |x|_p^{k_1 \cdot k_2} |1-x|_p^{k_1 \cdot k_3} \, dx, \tag{6}$$

where $k_i \in \mathbb{C}^d$, $\sum_{i=1}^4 k_i = 0$ and $k_i^2 = 2$. Adding the point ∞ does not change the value of the integral, but yields a compact domain of integration $\mathbb{P}^1(\mathbb{Q}_p)$. As in the previous section, we integrate over the space $\mathbb{P}^1$, but now view it as a moduli space:

$$\mathbb{P}^1 = \bar{M}_{0,4}, \tag{7}$$

the Deligne-Mumford compactification of the moduli space $M_{0,4}$ of 4-pointed projective lines. It is one-dimensional, because the first three punctures can be transformed to $\{0, 1, \infty\}$, whereas the fourth puncture runs through $\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. The boundary is given by letting λ run into $\{0, 1, \infty\}$. In order to also have 4 punctures in this case, one takes the singular curves as depicted in Figure 4. These are stable 4-pointed genus zero curves.

From (7), we can also write the 4-point amplitude (6) as

$$A_p^0(k_1, k_2, k_3, k_4) = \int_{\bar{M}_{0,4}} |x|^{k_1 \cdot k_2} |1-x|^{k_1 \cdot k_3} \, dx,$$

and consider now the contributions from different parts of the moduli space. By looking at the possible trees $\mathscr{T}(0, 1, \infty, \lambda)$ depicted in Figure 5, we see that the moduli space $M_{0,4}$ decomposes into 4 cells A, B, C, D. The first three cells which allow the edge length in the tropical curve to vary, are homeomorphic to the open unit interval, whereas cell D is 0-dimensional. The corresponding cell structure of the moduli space of tropical curves is illustrated in Figure 6.

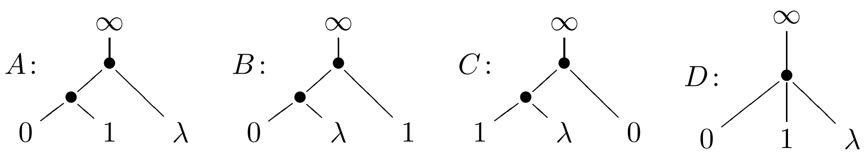

Figure 5: Trees representing the different cells of $M_{0,4}$.

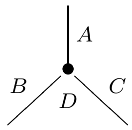

Figure 6: The cell structure of $M_{0,4}^{\mathrm{trop}}$.

Observe that D looks like a zero set. Indeed, similarly as in Section 2.5., we can find a measure on $\mathbb{P}^1 \cong \bar{M}_{0,4}$ which induces the uniform distribution $d\lambda$ on $\bar{M}_{0,4}^{\mathrm{trop}}$ via the tropicalisation map

$$\mathrm{trop}\colon \mathbb{P}^1 \to \bar{M}_{0,4}^{\mathrm{trop}}.$$

With $g(x) = |x|^{k_1 \cdot k_2} |1-x|^{k_1 k_3}$, it follows that

$$\int_D g c_1(\bar{O}(1)) = \int_D c_1(\bar{O}(1)) = \int_{\mathrm{trop}(D)} d\lambda = 0,$$

where the first equality[4] follows from $g|_D = 1$.

This approach generalists to the case of n points. Namely, assume we are given vectors $k_1, \dots, k_n \in \mathbb{C}^d$ with $\sum_{i=1}^n k_i = 0$ and $k_i^2 = 2$. Then we obtain:

Theorem 3..1 *The p-adic n-point tachyon string amplitude at the tree level*

$$A_p^0(k_1, \dots, k_n)$$
$$= \int_{\bar{M}_{0,n}} dx_2 \dots dx_{n-2} \prod_{i=2}^{n-2} |x_i|^{k_1 \cdot k_i} |1-x_i|^{k_{n-1} \cdot k_i} \prod_{2 \le i < j \le n-2} |x_i - x_j|^{k_i \cdot k_j}$$

[4]In [13], the notation $\int f c_1(\bar{O}(1))$ is preferred to our $\int g c_1(\bar{O}(1))$, where $f = -\log g$.

is contributed in the tropical limit only by those kinds of n-point configurations $\mathbb{P}^1 \setminus \{0, 1, x_2, \ldots, x_{n-2}, \infty\}$ for which $\mathscr{T}(0, 1, x_2, \ldots, x_{n-2}, \infty)$ is a binary tree.

4. Counting plane curves

Here, we sketch a construction of p-adic line bundles on the moduli space $\bar{M}_{g,n}$ of stable n-pointed genus g curves which can be used in order to count curves in the plane passing through given points satisfying some tangency conditions. Our approach is similar to the one in the previous section, except that now we consider the case in which the physics is "removed" from the problem.

4.1. p-adic ψ-classes?

The idea of ψ-classes is to allow the counting of curves with prescribed tangency conditions when passing through some prescribed subspaces of some target space.

Let L_i be the line bundle on $\bar{M}_{g,n}$ which yields in every curve C represented by $x = [C, p_1, \ldots, p_n] \in \bar{M}_{g,n}$ the cotangent in p_i. This is called the *i-th cotangent bundle* on $\bar{M}_{g,n}$. In complex algebraic geometry, the ψ-class ψ_i is then defined as a certain cohomology class called the *first Chern class* of L_i:

$$\psi_i := c_1(L_i) \in H^2(\bar{M}_{g,n}, \mathbb{Z})$$

which is nothing but the algebraic curvature encountered already in Section 2.5.. One obtains an intersection product

$$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle := \int_{\bar{M}_{g,n}} \psi^{k_1} \wedge \cdots \wedge \psi^{k_n} \tag{8}$$

which takes non-zero values if and only if $\sum k_i = 3g - 3 + n$.

The notation $\langle \tau_{k_1} \ldots \tau_{k_n} \rangle$ introduced by Witten [19] suggests a "physical" interpretation of the τ_{k_i} as operators on some Hilbert space whose correlator is the integral. In any case, the value of $\langle \tau_{k_1} \cdots \tau_{k_n} \rangle$ is symmetric in $k_1, \ldots, k_n \in \mathbb{N}$. The expression $\psi_1^{k_1} \wedge \cdots \wedge \psi_n^{k_n}$ can also be seen as a positive measure μ on the space $\bar{M}_{g,n}$ with total mass $\mu(\bar{M}_{g,n})$ given by (8).

Unfortunately, there is no sensible p-adic notion of Chern class of line bundles. The consequence is that there are no p-adic ψ-classes at hand. However, the p-adic analogon of the measure μ can be constructed as in Section 2.5.. Namely, take an O_K-model of $\bar{M}_{g,n}$ whose special fibre is a blow up of $\mathcal{M} := \bar{\mathcal{M}}_{g,n} \otimes \kappa$ in the boundary in such a way that the vertices of $\bar{M}_{g,n}^{\mathrm{trop}}$ correspond to the irreducible components of $\mathcal{M}$. Take an O_K-model $\mathscr{L}_i$ of L_i. Then (5) defines a measure

$$c_1(\bar{L}_1)^{k_1} \wedge \cdots \wedge c_1(\bar{L}_n)^{k_n}$$

on $\bar{M}_{g,n}$ which is supported on the points above the generic points of the components of $\mathcal{M}$. A smoothing process yields as in Section 2.5. a measure μ_p for which $\mathrm{trop}_*(\mu_p)$ is a piece-wise Haar measure on $\bar{M}_{g,n}^{\mathrm{trop}}$ [3]. It is uniform on the closure of each maximal cell of $\bar{M}_{g,n}^{\mathrm{trop}}$. Now, each cell parametrises tropical curves of fixed combinatorial type, and the maximal cells correspond to trivalent semi-graphs. So, we state our result:

Theorem 4..1 *The p-adic "correlator"*

$$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle := \int_{\bar{M}_{g,n}} \mu_p$$

is contributed only by the locus of trivalent Mumford curves in $\bar{M}_{g,n}$, and is a weighted graph sum.

The proof follows by observing that $\mathrm{trop}_*(\mu_p)$ is a measure for which all cells of dimension lower than $3g-3+n$ are zero sets.

4.2. Including "gravity"

We now consider the problem of counting curves of degree d and genus g passing through $n = 3d + g - 1$ points in the plane. A solution to this problem was predicted through Witten's conjecture [19], proved by Kontsevich [14]. The idea is to count maps $C \to \mathbb{P}^2$ of n-pointed curves into the plane, called "instantons". The existence of a target space X (here: $\mathbb{P}^2$) introduces "gravity" to the system. By using so-called *stable maps*, one obtains a compactification of the moduli space of instantons. The theory then allows the construction of "gravitational" ψ-classes and correlators.

Recently, it was shown that counting maps of tropical curves to the tropical plane $\mathbb{TP}^2$ yields the same numbers as for usual curves [15, 11]. Those numbers are also called *Gromov-Witten invariants*. From a p-adic point of view, the correspondence between the classical and tropical Gromov-Witten numbers does not come as a surprise. Namely, we have a commuting diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\mathrm{trop}"'] & \mathbb{P}^2 \arrow[d, "\mathrm{Val}_{\mathfrak{A}}"] \\ \Gamma \arrow[r] & \mathbb{TP}^2 \end{tikzcd}$$

with a lot of choices of maps $\mathrm{Val}_{\mathfrak{A}} \colon \mathbb{P}^2 \to \mathbb{TP}^2$. Namely, for any configuration $\mathfrak{A}$ of three lines in $\mathbb{P}^2$ in general position, there is a transformation $\alpha \in \mathrm{PGL}_3(K)$ which takes $\mathfrak{A}$ to the three standard lines, i.e. the 2 coordinate lines in K^2 and the line at infinity. Then define

$$\mathrm{Val}_{\mathfrak{A}} := \mathrm{Val} \circ \alpha.$$

Classically, the number of curves passing through a set $\mathscr{P}$ of $n = 3d+g-1$ points in $\mathbb{P}^2$ does not depend on the position of the points, as long as they are in general position. It follows that if for some $\mathfrak{A}$, the set $\mathrm{Val}_{\mathfrak{A}}(\mathscr{P})$ consists of n points in $\mathbb{TP}^2$ tropically in general position, then the number of tropical curves Γ with $b_1(\Gamma) = g$ and degree d passing through $\mathrm{Val}_{\mathfrak{A}}(\mathscr{P})$ does not depend on their positions in $\mathbb{TP}^2$. As a side effect of this observation, we obtain the result:

Theorem 4..2 *If there exists a line configuration $\mathfrak{A}$ such that $\mathrm{Val}_{\mathfrak{A}}(\mathscr{P})$ consists of n points tropically in general position, then*

$$N_{d,g}^{\mathrm{Mumf}}(\mathscr{P}) = N_{d,g}.$$

i.e. the plane curves of degree d and genus g passing through $\mathscr{P}$ are all Mumford curves.

4.3. The CMZ-conjecture in adelic string theory

A crucial observation in Section 3. was that integrating over all n-point configurations on the projective line means in fact integration over the moduli space $M_{0,n}$ of n-pointed genus 0 curves. Hence, a straightforward generalisation to the multiloop case means to integrate over $M_{g,n}$, the moduli space of n-pointed genus g curves, resp. its Deligne-Mumford compactification $\bar{M}_{g,n}$. Indeed, Chekhov et al. [6] describe a p-adic multiloop amplitude. However, in their calculations they vary only the n points on the p-adic Riemann surface X while keeping the surface itself fixed. But their conjecture [6, Conj. 4.3.3] is a statement about the amplitude when both, the points and the holomorphic structure on X, vary. Tropically, this amounts to varying the possible combinatorial types of $\Gamma = \operatorname{trop} X$ as well as the possible lengths of the bounded edges of Γ. Hence, we can formulate:

Conjecture 4..3 *The p-adic string amplitude*

$$A_p^g(k_1, \ldots, k_n) = \int_{\bar{M}_{g,n}^{\mathrm{trop}}} \operatorname{trop}_* \mu_p$$

is contributed in the tropical limit by precisely the binary Mumford curves via weighted summation of the graphs underlying their tropicalisations.

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