On certain classes of algebraic curvature tensors*

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Abstract

In this text we deal with Rakić duality principle. We search for a connection between Osserman and Rakić curvature tensor. We prove that 3-dimensional Rakić is Osserman. We investigate Rakić duality using Fiedler's skew-symmetric decomposition, and prove that Osserman curvature tensor with constant Fiedler's signs is Rakić.

1. Introduction

Let us begin with the basic notation and terminology which are used throughout this work. Let R be an algebraic curvature tensor on a vector space $\mathcal{V}$ equipped with an indefinite metric g of the signature $(\nu, n-\nu)$. The sign $\varepsilon_X = g(X,X)$ denotes the norm of $X \in \mathcal{V}$, and it determines various types of vectors. We say that $X \in \mathcal{V}$ is timelike (if $\varepsilon_X < 0$), spacelike ($\varepsilon_X > 0$), null ($\varepsilon_X = 0$), nonnull ($\varepsilon_X \neq 0$), or unit ($\varepsilon_X \in \{-1,1\}$). The curvature operator $\mathcal{R}$ is connected with R via equation $R(X,Y,Z,W) = g(\mathcal{R}(X,Y)Z,W)$. If $(E_1, E_2, ..., E_n)$ is an orthonormal basis of $\mathcal{V}$, then we use short notations $\varepsilon_i = \varepsilon_{E_i}$ and $R_{ijkl} = R(E_i,E_j,E_k,E_l)$. For the initial definitions and deeper explanations of this topic, the reader can consult Gilkey's books [9, 10].

The Jacobi operator $\mathcal{J}_X : \mathcal{V} \to \mathcal{V}$ is a natural operator defined by $\mathcal{J}_X(Z) = \mathcal{R}(Z,X)(X)$ for all $Z \in \mathcal{V}$. In the case of nonnull $X \in \mathcal{V}$, $\mathcal{J}_X$ preserves nondegenerate hyperspace $\{X\}^\perp = \{Y \in \mathcal{V} : X \perp Y\}$, and we have the reduced Jacobi operator $\tilde{\mathcal{J}}_X : \{X\}^\perp \to \{X\}^\perp$, given by $\tilde{\mathcal{J}}_X = \mathcal{J}_X|_{\{X\}^\perp}$.

We say that R is an Osserman curvature tensor if the characteristic polynomial of $\mathcal{J}_X$ is constant on both pseudo-spheres, in particular on positive ($\varepsilon_X = 1$) and negative ($\varepsilon_X = -1$) one. In a pseudo-Riemannian setting,

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Jordan normal form plays a crucial role, since characteristic polynomial does not determine the eigen-structure of a symmetric linear operator. We say that R is a Jordan Osserman curvature tensor if the Jordan normal form of $\mathcal{J}_X$ is constant on both pseudo-spheres. An Osserman curvature tensor, whose Jacobi operator $\mathcal{J}_X$ is diagonalizable for all nonnull X, we call diagonalizable Osserman.

In the Riemannian setting ($\nu = 0$), it is known that a local two-point homogeneous space (flat or locally rank one symmetric space) has a constant characteristic polynomial on the unit sphere bundle. Osserman wondered if the converse held [15], and this question has been called the Osserman conjecture by subsequent authors. During the solution of some particular cases of the conjecture, the implication

$$\mathcal{J}_X(Y) = \lambda Y \Rightarrow \mathcal{J}_Y(X) = \lambda X \tag{1}$$

appeared naturally, and if it holds, it can significantly simplify some calculations. The first results in this topic are given by Chi [7], who proved the conjecture in the cases of dimensions $n \neq 4k$, $k > 1$. In his work he used the fact that (1) holds for extremal eigenvalues λ of the Jacobi operator. Rakić [16] proved the correctness of (1) for every eigenvalue λ of the Jacobi operator, and this statement has been called Rakić duality principle [9]. After that, Rakić duality is reproved by Gilkey [9], and it become a beneficial tool for the solution of the conjecture. Moreover, the best results in this topic gave Nikolayevsky [12, 13, 14], who used Rakić duality [13] to prove Osserman conjecture in all dimensions, except some possibilities in dimension $n = 16$.

The variant of the Osserman conjecture has been appeared in a pseudo-Riemannian setting. For example, in the Lorentzian setting ($\nu = 1$), an Osserman manifold necessarily has a constant sectional curvature [5]. Observation of Osserman manifolds in the signature $(2, 2)$ become very popular, and it is worth noting results from [6], which are based on the discussion of possible Jordan normal forms of the Jacobi operator.

This is why we start to investigate the duality principle for Osserman curvature tensor in a pseudo-Riemannian setting. In a pseudo-Riemannian setting, the implication (1) looks inaccurate, and therefore we corrected it in the following way [1, 4].

Definition 1 (Rakić duality) *We say that a curvature tensor R satisfies Rakić duality for the value λ, if for all mutually orthogonal units $X, Y \in \mathcal{V}$ holds*

$$\mathcal{J}_X(Y) = \varepsilon_X \lambda Y \Rightarrow \mathcal{J}_Y(X) = \varepsilon_Y \lambda X. \tag{2}$$

We say that R is Rakić if it satisfies Rakić duality for all $\lambda \in \mathbb{R}$.

The Rakić duality for Osserman curvature tensor works nicely for every known example, which motivated us to post the following conjecture.

Conjecture 1 *Osserman pseudo-Riemannian curvature tensor is Rakić.*

Unfortunately we failed to prove this conjecture in general. In our previous work we gave the affirmative answer only for the conditions of small index ($\nu \leq 1$) [1, 4], low dimension ($n \leq 4$) [1, 4, 3], and some possibilities with small numbers of eigenvalues of the reduced Jacobi operator [2]. Seeing that Rakić property is natural for Osserman curvature tensor (at least in the Riemannian setting), one can ask if the converse held.

Conjecture 2 *Rakić pseudo-Riemannian curvature tensor is Osserman.*

2. Three-dimensional case

In this section we deal with three-dimensional Rakić pseudo-Riemannian curvature tensor. Let us start with the following universal lemma.

Lemma 1 *If $\mathcal{J}_X(Y) = \varepsilon_X \lambda Y$ and $\mathcal{J}_Y(X) = \varepsilon_Y \lambda X$, then for all $\alpha, \beta \in \mathbb{R}$ holds $\mathcal{J}_{\alpha X+\beta Y}(\varepsilon_Y \beta X - \varepsilon_X \alpha Y) = \varepsilon_{\alpha X+\beta Y}\lambda(\varepsilon_Y \beta X - \varepsilon_X \alpha Y)$.*

Proof. This lemma is a consequence of the straightforward calculations.

$$\begin{aligned}
&\mathcal{J}_{\alpha X+\beta Y}(\varepsilon_Y \beta X - \varepsilon_X \alpha Y)\\
&\qquad = \mathcal{R}(\varepsilon_Y \beta X - \varepsilon_X \alpha Y, \alpha X + \beta Y)(\alpha X + \beta Y)\\
&\qquad = \mathcal{R}(\varepsilon_Y \beta X, \beta Y)(\alpha X + \beta Y) + \mathcal{R}(-\varepsilon_X \alpha Y, \alpha X)(\alpha X + \beta Y)\\
&\qquad = -\varepsilon_Y \alpha \beta^2 \mathcal{J}_X(Y) + \varepsilon_Y \beta^3 \mathcal{J}_Y(X) - \varepsilon_X \alpha^3 \mathcal{J}_X(Y) + \varepsilon_X \alpha^2 \beta \mathcal{J}_Y(X)\\
&\qquad = \beta(\varepsilon_X \alpha^2 + \varepsilon_Y \beta^2)\mathcal{J}_Y(X) - \alpha(\varepsilon_X \alpha^2 + \varepsilon_Y \beta^2)\mathcal{J}_X(Y)\\
&\qquad = (\varepsilon_X \alpha^2 + \varepsilon_Y \beta^2)(\beta \varepsilon_Y \lambda X - \alpha \varepsilon_X \lambda Y)\\
&\qquad = \varepsilon_{\alpha X+\beta Y}\lambda(\varepsilon_Y \beta X - \varepsilon_X \alpha Y)
\end{aligned}$$

□

Let us investigate Conjecture 2 for low dimension $n = 3$. It is known that three-dimensional Einstein (consequently it holds for Osserman) curvature tensor necessarily has constant sectional curvature.

Theorem 1 *Three-dimensional Rakić curvature tensor is of constant sectional curvature.*

Proof. In order to apply Rakić property we need to show that there is a pair (X, Y) of mutually orthogonal units, where Y is an eigenvector of $\mathcal{J}_X$. Let (E_1, E_2, E_3) be an arbitrary orthonormal basis of $\mathcal{V}$, such that $\varepsilon_1 = \varepsilon_2$. The matrix of the Jacobi operator $\mathcal{J}_{E_3}$ is

$$\mathcal{J}_{E_3} = \begin{pmatrix} \varepsilon_1 R_{1331} & \varepsilon_1 R_{2331} & 0 \\ \varepsilon_2 R_{1332} & \varepsilon_2 R_{2332} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and therefore its reduced Jacobi operator $\tilde{\mathcal{J}}_{E_3}$ has characteristic polynomial

$$x^2 - (\varepsilon_1 R_{1331} + \varepsilon_2 R_{2332})x + \varepsilon_1 \varepsilon_2 R_{1331} R_{2332} - \varepsilon_1 \varepsilon_2 R_{2331} R_{1332} = 0.$$

Let D be a discriminant of the previous quadratic equation, then

$$D = (\varepsilon_1 R_{1331} + \varepsilon_2 R_{2332})^2 - 4\varepsilon_1\varepsilon_2 R_{1331} R_{2332} + 4\varepsilon_1\varepsilon_2 (R_{1332})^2.$$

Because of $\varepsilon_1 = \varepsilon_2$, we have $\varepsilon_1\varepsilon_2 = (\varepsilon_1)^2 = 1$, and thus

$$D = (\varepsilon_1 R_{1331} - \varepsilon_2 R_{2332})^2 + 4(R_{1332})^2 \geq 0.$$

If $D > 0$ then our quadratic equation has two distinct real roots, which represents two distinct eigenvalues of $\tilde{\mathcal{J}}_{E_3}$, and hence there are two (nondegenerate) one-dimensional eigenspaces. Otherwise, $D = 0$ gives $\varepsilon_1 R_{1331} = \varepsilon_2 R_{2332}$ and $R_{1332} = 0$. Thus $\tilde{\mathcal{J}}_{E_3}$ is diagonal with double root which is associated with a two-dimensional eigenspace.

The previous text proved that there exists an orthonormal basis (X, Y, Z), such that Y and Z are eigenvectors of $\mathcal{J}_X$. Consequently, by Rakić duality, X is an eigenvector of $\mathcal{J}_Y$. The conditions of Lemma 1 hold, and hence $\varepsilon_Y \beta X - \varepsilon_X \alpha Y$ is an eigenvector of $\mathcal{J}_{\alpha X + \beta Y}$. Moreover, for $\alpha, \beta \in \mathbb{R}$ such that $\alpha^2\varepsilon_X + \beta^2\varepsilon_Y \neq 0$, both $\alpha X + \beta Y$ and $\varepsilon_Y \beta X - \varepsilon_X \alpha Y$ are mutually orthogonal and nonnull. The Jacobi operator is symmetric linear operator and therefore

$$g(\mathcal{J}_{\alpha X+\beta Y}(Z), \varepsilon_Y \beta X - \varepsilon_X \alpha Y) = g(Z, \mathcal{J}_{\alpha X+\beta Y}(\varepsilon_Y \beta X - \varepsilon_X \alpha Y)) = 0.$$

Hence $\mathcal{J}_{\alpha X+\beta Y}(Z)$ is orthogonal to both $\varepsilon_Y \beta X - \varepsilon_X \alpha Y$ and $\alpha X + \beta Y$, which gives $\mathcal{J}_{\alpha X+\beta Y}(Z) \perp \mathrm{Span}\{X, Y\} = \mathrm{Span}\{Z\}^\perp$, and consequently Z is an eigenvector of $\mathcal{J}_{\alpha X+\beta Y}$. According to Rakić duality, $\alpha X + \beta Y$ is an eigenvector of $\mathcal{J}_Z$, which is possible only if $\mathrm{Span}\{X, Y\}$ is a two-dimensional eigenspace of $\tilde{\mathcal{J}}_Z$. Similarly one can prove that $\mathrm{Span}\{X, Z\}$ is an eigenspace of $\tilde{\mathcal{J}}_Y$. Thus arise

$$\frac{R(X,Y,Y,X)}{\varepsilon_X\varepsilon_Y} = \frac{R(Y,Z,Z,Y)}{\varepsilon_Y\varepsilon_Z} = \frac{R(X,Z,Z,X)}{\varepsilon_X\varepsilon_Z} = \kappa,$$

and $R(Y,X,X,Z) = R(X,Y,Y,Z) = R(X,Z,Z,Y) = 0$, which obviously completely describes R, and therefore R has constant sectional curvature κ. $\square$

3. Rakić duality and Fiedler's tensor

In this section we try to describe Rakić duality property. Let $(E_1, ..., E_n)$ be an arbitrary orthonormal basis of the vector space $\mathcal{V}$ of the signature $(\nu, n-\nu)$. Let us start with the left hand side of the equation (2),

$$\mathcal{J}_X(Y) = \varepsilon_X \lambda Y. \tag{3}$$

The equation (3) means that Y is an eigenvector of $\mathcal{J}_X$ for the eigenvalue $\varepsilon_X\lambda$. The Jacobi operator can be expressed using the curvature tensor on the following way

$$\mathcal{J}_X(Y)=\mathcal{R}(Y,X)X=\sum_l \varepsilon_l R(Y,X,X,E_l)E_l.$$

If we set nonnull $X=\sum_i \alpha_i E_i$ and $Y=\sum_i \beta_i E_i$, the equation (3) become

$$\sum_{i,j,k,l}\varepsilon_l\beta_i\alpha_j\alpha_k R_{ijkl}E_l=\varepsilon_X\lambda\sum_l \beta_l E_l,$$

and finally

$$(\forall l)\ \sum_{i,j,k}\varepsilon_l\beta_i\alpha_j\alpha_k R_{ijkl}=\varepsilon_X\lambda\beta_l.$$

According to work of Fiedler [8], and later development by Gilkey [8, 9], for every algebraic curvature tensor R, there exist finite numbers of skew-symmetric tensors Ω of order 2 (i.e. the coordinates of Ω satisfy $\Omega_{ij}=-\Omega_{ji}$), such that R has a representation

$$R_{ijkl}=\sum_\Omega \varepsilon_\Omega\frac{1}{3}\left(2\Omega_{ij}\Omega_{kl}+\Omega_{ik}\Omega_{jl}-\Omega_{il}\Omega_{jk}\right),$$

with $\varepsilon_\Omega\in\{-1,1\}$. Using this fact the equation (3) is equivalent to

$$(\forall l)\ \sum_\Omega\sum_{i,j,k}\frac{1}{3}\varepsilon_\Omega\varepsilon_l\beta_i\alpha_j\alpha_k\left(2\Omega_{ij}\Omega_{kl}+\Omega_{ik}\Omega_{jl}-\Omega_{il}\Omega_{jk}\right)=\varepsilon_X\lambda\beta_l.$$

We can simplify the previous formula using the symmetry by j and k.

$$\sum_{i,j,k}\beta_i\alpha_j\alpha_k\Omega_{ik}\Omega_{jl}=\sum_{i,k,j}\beta_i\alpha_k\alpha_j\Omega_{ij}\Omega_{kl}=\sum_{i,j,k}\beta_i\alpha_j\alpha_k\Omega_{ij}\Omega_{kl}$$

$$\sum_{i,j,k}\beta_i\alpha_j\alpha_k\Omega_{il}\Omega_{jk}=\sum_{i,k,j}\beta_i\alpha_k\alpha_j\Omega_{il}\Omega_{kj}=-\sum_{i,j,k}\beta_i\alpha_j\alpha_k\Omega_{il}\Omega_{jk}=0$$

Therefore the equation (3) is equivalent to

$$(\forall l)\ \sum_\Omega\sum_{i,j,k}\varepsilon_\Omega\varepsilon_l\beta_i\alpha_j\alpha_k\Omega_{ij}\Omega_{kl}=\varepsilon_X\lambda\beta_l.$$

Let us split sums on the left hand side

$$(\forall l)\ \sum_\Omega\varepsilon_\Omega\varepsilon_l\sum_{i,j}\alpha_i\beta_j\Omega_{ij}\sum_k\alpha_k\Omega_{kl}=-\varepsilon_X\lambda\beta_l.$$

If we introduce the short notation

$$\Theta^{\Omega}_{PQ} = \sum_{i,j} \mu_i \nu_j \Omega_{ij},$$

for $P = \sum_i \mu_i E_i$ and $Q = \sum_j \nu_j E_j$, the equation (3) become equivalent to

$$(\forall l)\ \sum_{\Omega} \varepsilon_\Omega \Theta^{\Omega}_{XY} \Theta^{\Omega}_{XE_l} = -\varepsilon_l \varepsilon_X \lambda \beta_l. \tag{4}$$

Using (3) $\Leftrightarrow$ (4) and $\Theta^{\Omega}_{YX} = -\Theta^{\Omega}_{XY}$, we get the equivalent form of Rakić duality condition (2)

$$(\forall l)\ \sum_{\Omega} \varepsilon_\Omega \Theta^{\Omega}_{XY} \Theta^{\Omega}_{XE_l} = -\varepsilon_l \varepsilon_X \lambda \beta_l \ \Rightarrow\ (\forall l)\ \sum_{\Omega} \varepsilon_\Omega \Theta^{\Omega}_{XY} \Theta^{\Omega}_{YE_l} = \varepsilon_l \varepsilon_Y \lambda \alpha_l. \tag{5}$$

Let us sum all n equations ($1 \le l \le n$) from (4) multiplied by β_l

$$\sum_l \beta_l \sum_{\Omega} \varepsilon_\Omega \Theta^{\Omega}_{XY} \Theta^{\Omega}_{XE_l} = -\sum_l \beta_l \varepsilon_l \varepsilon_X \lambda \beta_l,$$

After the substitutions $\sum_l \beta_l \Theta^{\Omega}_{XE_l} = \Theta^{\Omega}_{XY}$ and $\sum_l \varepsilon_l \beta_l^2 = \varepsilon_Y$ we get important equation.

$$\sum_{\Omega} \varepsilon_\Omega (\Theta^{\Omega}_{XY})^2 = -\varepsilon_X \varepsilon_Y \lambda. \tag{6}$$

Let us stop here to notice the following interesting statements in the case when Fiedler's terms have a constant signs, i.e. $\varepsilon_\Omega = \text{const}$.

Theorem 2 *If R is a curvature tensor with $\varepsilon_\Omega = \text{const}$, then it satisfies Rakić duality for the value* 0.

Proof. If the sign ε_Ω is constant for all skew-symmetric tensors Ω in the sense of Fiedler, then the equation (6) gives

$$0 \le \sum_{\Omega} (\Theta^{\Omega}_{XY})^2 = -\varepsilon_\Omega \varepsilon_X \varepsilon_Y \lambda.$$

The value $\lambda = 0$ gives $\sum_\Omega (\Theta^{\Omega}_{XY})^2 = 0$ and therefore $\Theta^{\Omega}_{XY} = 0$ for all Ω. The formula (5) obviously holds and R satisfies Rakić duality for the value 0. $\square$

Theorem 3 *A diagonalizable Osserman curvature tensor R with $\varepsilon_\Omega = \text{const}$ is Rakić.*

Proof. Like the previous proof, the equation (6) with $\varepsilon_\Omega = \text{const}$ gives $0 \leq -\varepsilon_\Omega \varepsilon_X \varepsilon_Y \lambda$. Since by Theorem 2, R satisfies Rakić duality for the value 0, we set $\lambda \neq 0$, and therefore $\varepsilon_\Omega \varepsilon_X \varepsilon_Y \lambda < 0$. It implies the constant sign of ε_Y, which proves that an eigenspace of $\mathcal{J}_X$ for an eigenvalue $\varepsilon_X \lambda$ has the same type of vectors. Especially, there are no nonzero null vectors in the eigenspace of $\mathcal{J}_X$ for an eigenvalue $\varepsilon_X \lambda$. According to our previous work [1, 4], diagonalizable Osserman curvature tensor, such that $\mathcal{J}_X$ has no null eigenvector for an eigenvalue $\varepsilon_X \lambda$, satisfies the duality principle for the value λ, which completes the proof. □

The diagonalizability from Theorem 3 is a natural condition. Let us remark that, according to Gilkey and Ivanova [11], Jordan Osserman curvature tensor of a non-balanced signature ($n \neq 2\nu$) is necessarily diagonalizable.

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