Killing-Yano tensors, Dirac-type operators and quantum anomalies *

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Abstract

We study the Dirac equation on curved spaces with symmetries pointing out the role of the Killing-Yano tensors in the construction of the Dirac-type operators. In particular the covariantly constant Killing-Yano tensors realize a certain square root of the metric tensor and produce Dirac-type operators commuting with the standard Dirac one. The general results are applied to the case of the four-dimensional Euclidean Taub-Newman-Unti-Tamburino space and the Minkowski spacetime. The gravitational and axial anomalies are studied for generalized Euclidean Taub-NUT metrics which admit hidden symmetries analogous to the Runge-Lenz vector of the Kepler-type problem. Using the Atiyah-Patodi-Singer index theorem for manifolds with boundaries, it is shown that these metrics make no contribution to the axial anomaly.

1. Introduction

In order to study the geodesic motions and the conserved classical and quantum quantities for fermions on curved spaces, the symmetries of the backgrounds proved to be very important. We mention that the following two generalization of the Killing (K) vector equation have become of interest in physics:

1. A symmetric tensor field $K_{\mu_1\dots\mu_r}$ is called a Stäckel-Killing (S-K) tensor of valence r if and only if

$$K_{(\mu_1\dots\mu_r;\lambda)} = 0. \tag{1}$$

   The usual Killing (K) vectors correspond to valence $r = 1$ while the hidden symmetries are encapsulated in S-K tensors of valence $r > 1$.

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2. A tensor $f_{\mu_1 \ldots \mu_r}$ is called a Killing-Yano (K-Y) [1] tensor of valence r if it is totally antisymmetric and it satisfies the equation

$$f_{\mu_1 \ldots (\mu_r;\lambda)} = 0. \tag{2}$$

These objects can be characterized in several equivalent ways. For example, K-Y tensors can be defined as differential forms on a manifold whose covariant derivative is totally antisymmetric. A symmetric tensor $K_{\alpha_1 \ldots \alpha_r}$ on a manifold M is a Killing tensor iff the quantity $K_{\alpha_1 \ldots \alpha_r} \dot{c}^{\alpha_1} \ldots \dot{c}^{\alpha_r}$ is constant along every geodesic c in M.

The K-Y tensors play an important role in theories with spin and especially in the Dirac theory on curved spacetimes where they produce first order differential operators, called Dirac-type operators, which anticommute with the standard Dirac one, D_s [2]. Another virtue of the K-Y tensors is that they enter as square roots in the structure of several second rank S-K tensors that generate conserved quantities in classical mechanics or conserved operators which commute with D_s. The construction of Carter and McLenaghan depended upon the remarkable fact that the (symmetric) S-K tensor $K_{\mu\nu}$ involved in the constant of motion quadratic in the four-momentum p_μ

$$Z = \frac{1}{2}\, K^{\mu\nu} p_\mu p_\nu \tag{3}$$

has a certain square root in terms of K-Y tensors $f_{\mu\nu}$:

$$K_{\mu\nu} = f_{\mu\lambda} f^{\lambda}_{\ \nu}\ . \tag{4}$$

These attributes of the K-Y tensors lead to an efficient mechanism of supersymmetry especially when the S-K tensor $K_{\mu\nu}$ in eq. (3) is proportional with the metric tensor $g_{\mu\nu}$ and the corresponding K-Y tensors in eq. (4) are covariantly constant. Then each tensor of this type, f^i, gives rise to a Dirac-type operator, D^i, representing a supercharge of the superalgebra $\{D^i, D^j\} \propto D_s^2 \delta_{ij}$.

The index of the Dirac operator is a useful tool to investigate the topological properties of the manifold as well as in computing axial quantum anomalies in field theories. In even-dimensional spaces one can define the index of a Dirac operator as the difference between the number of linearly independent zero modes with eigenvalues $+1$ and -1 under γ_5. A remarkable result states the equality of the indices of the standard and non-standard Dirac operators [3].

The general results are applied to the case of the four-dimensional Euclidean Taub-Newman-Unti-Tamburino (Taub-NUT) space and its generalization [4]. The Taub-NUT family of metrics is involved in many modern studies in physics like gravitational instantons, monopoles, strings, membranes, etc.

For the generalized Taub-NUT spaces, in [5] we computed the gravitational and axial quantum anomaly, interpreted as the index of the Dirac operator of these metrics, on annular domains and on disks, with the non-local Atiyah-Patodi-Singer boundary condition.

The structure of the paper is as follows: We start the next section with a description of the Dirac equation on a curved background with symmetries. In the next two sections we study the Dirac-type operators associated to covariantly constant K-Y tensors. In Section 5 we investigate the gravitational anomalies pointing out the role of the K-Y tensors. In Section 6 we present the axial anomalies for generalized Taub-NUT space. In two appendices the general results are applied to the Euclidean Taub-NUT and Minkowski spaces.

2. Dirac equation on a curved background

In what follows we shall consider the Dirac operator on a curved background which has the form

$$D_s = i\gamma^\mu \hat{\nabla}_\mu . \tag{5}$$

In this expression the Dirac matrices γ_μ are defined in local coordinates by the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu} I$ and $\hat{\nabla}_\mu$ denotes the canonical covariant derivative for spinors. The essential properties of this covariant derivative are summarized in the following equations

$$\begin{aligned} &\hat{\nabla}_\mu \gamma^\mu = 0, \\ &\hat{\nabla}_{[\rho}\hat{\nabla}_{\mu]} = \frac{1}{4}\, R_{\alpha\beta\rho\mu}\gamma^\alpha\gamma^\beta , \end{aligned}$$

where $R_{\alpha\beta\rho\mu}$ denotes the components of the Riemann curvature tensor.

Carter and McLenaghan showed that in the theory of Dirac fermions for any isometry with K vector R_μ there is an appropriate operator [2]:

$$X_k = i(R^\mu \hat{\nabla}_\mu - \frac{1}{4}\, \gamma^\mu \gamma^\nu R_{\mu;\nu})$$

which commutes with the *standard* Dirac operator (5).

Moreover each K-Y tensor $f_{\mu\nu}$ produces a *non-standard* Dirac operator of the form

$$D_f = i\gamma^\mu (f_\mu{}^\nu \hat{\nabla}_\nu - \frac{1}{6}\, \gamma^\nu \gamma^\rho f_{\mu\nu;\rho}) \tag{6}$$

which anticommutes with the standard Dirac operator D_s.

3. Roots and Dirac-type operators

Let us consider a manifold M_n of dimension n with the metric $g_{\mu\nu}$.

Definition 1 *The non-singular real or complex-valued K-Y tensor f of rank 2 defined on M_n which satisfies*

$$f^{\mu\cdot}_{\cdot\alpha} f_{\mu\beta} = g_{\alpha\beta} , \tag{7}$$

is called an unit root of the metric tensor of M_n, or simply an unit root of M_n.
Any K-Y tensor that satisfy eq. (7) is covariantly constant [6, 7], i.e.,

$$f_{\mu\nu;\sigma} = 0\,. \tag{8}$$

Since eq. (7) can be written as $f^{\mu\cdot}_{\cdot\alpha} f^{\alpha\cdot}_{\cdot\nu} = -\delta^{\mu}_{\nu}$ this takes the matrix form,

$$\langle f \rangle^2 = -\mathbf{I}\,, \tag{9}$$

where the notation $\mathbf{I}$ stands for the $n \times n$ identity matrix. Hereby we see that the unit roots are matrix representations of several *complex units* (similar to $i \in \mathbb{C}$) with usual properties as, for example, $\langle f \rangle^{-1} = -\langle f \rangle$. The unit roots having only *real*-valued components are called *complex structures* and represent automorphisms of the tangent fiber bundle $\mathcal{T}(M_n)$ of M_n. The manifold possessing such structures are said to have a Kählerian geometry. However, the unit roots considered here are beyond this case since these are defined as automorphisms of the complexified fiber bundle $\mathcal{T}(M_n) \otimes \mathbb{C}$.

As in the case of the complex structures of the Kählerian geometries, the matrices of the unit roots have specific algebraic properties resulted from eq. (9).

Lemma 1 *The matrix of any root of M_n is equivalent with a matrix completely reducible in 2×2 diagonal blocks.*

Proof. The matrix $\langle f \rangle$ which satisfies eq. (9) has only two-dimensional invariant subspaces spanned by pairs of vectors z and $\langle f \rangle z$. On these subspaces, eq. (9) is solved in local frames by two types of 2×2 unimodular blocks without diagonal elements: either skew-symmetric blocks with factors ± 1, when the involved dimensions are of the same signature, or symmetric ones with pure imaginary phase factors, $\pm i$, if the signatures are opposite. However, the diagonalization procedure can not be continued using transformations of the gauge group of the metric since these preserve the form of the 2×2 blocks which are proportional with the generators of the subgroups $SO(2)$ or $SO(1,1)$ acting on the corresponding invariant subspaces. □

This selects the geometries allowing unit roots.

Corollary 1 *The unit roots are allowed only by manifolds M_n with an even number of dimensions, $n = 2k$, $k \le l$.*

Proof. If n is odd then the 2×2 blocks do not cover all dimensions, so that $\det\langle f \rangle = 0$ and f is no more an unit root of M_n. □

Corollary 2 *The unit roots of M_n have real matrices only when the metric η has a signature with even n_+ and n_-. Otherwise the unit roots have only complex-valued matrices. In both cases the matrices of the unit roots are unimodular, i.e.* $\det\langle f \rangle = 1$.

It is clear that for the real-valued unit roots (i.e., complex structures) one can construct the *symplectic* 2-forms $\tilde{\omega} = \frac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu$ which are closed and non-degenerate.

The above properties indicate that the unit roots are defined up to sign. Therefore, if two unit roots f_1 and f_2 do not obey the condition $f_1 = \pm f_2$ then these will be considered *different* between themselves. We denote by $\mathcal{R}_1(M_n)$ the set of all different unit roots of the manifold M_n. On the other hand, when an unit root f is multiplied by an arbitrary *real* number $\alpha \neq 0$, we say that $\xi(x) = \alpha f(x)$ is a *root* of norm $\|\xi\| = |\alpha|$. Thus we can associate to any unit root f the one-dimensional linear real space $L_f = \{\xi \,|\, \xi = \alpha f,\, \alpha \in \mathbb{R}\}$ in which each non vanishing element is a root. According to Corollary 2, when the metric η is pseudo-Euclidean, the unit roots can have complex matrix elements and in that case the unit root f and its *adjoint*, f^*, are different. This last one generates its own linear real space L_{f^*} of adjoint roots which satisfy $[\,\langle\xi\rangle^*\,,\,\langle\xi'\rangle\,] = 0$, $\forall \xi,\, \xi' \in L_f$ since the matrices of f and f^* commutes with each other, having same diagonal blocks up to signs.

The K-Y tensor gives rise to Dirac-type operators of the form (6). These have an important property formulated in the next theorem [6].

Theorem 1 *The Dirac-type operator D_f produced by the K-Y tensor f satisfies the condition*

$$(D_f)^2 = D^2 \tag{10}$$

if and only if f is an unit root.

Proof. The arguments of Ref. [6] show that the condition eq. (10) is equivalent with eqs. (7) and (8). Moreover, we note that for $f \in \mathcal{R}_1(M_n)$ the square of the Dirac-type operator

$$D_f = i f_{\mu\cdot}^{\cdot\,\nu} \gamma^\mu \nabla_\nu \tag{11}$$

has to be calculated exploiting the identity

$$0 = f_{\mu\nu;\alpha;\beta} - f_{\mu\nu;\beta;\alpha} = f_{\mu\sigma} R^{\sigma}_{\cdot\,\nu\alpha\beta} + f_{\sigma\nu} R^{\sigma}_{\cdot\,\mu\alpha\beta},$$

which gives

$$R_{\mu\nu\alpha\beta} f^{\mu\,\cdot}_{\cdot\,\sigma} f^{\nu\,\cdot}_{\cdot\,\tau} = R_{\sigma\tau\alpha\beta}$$

and leads to eq. (10). □

Thus we conclude that the equivalence of the condition (10) with eqs. (7) and (8) holds in any geometry of dimension $n = 2k$ allowing roots. When $f^* \neq f$ then D_{f^*} is different from D_f even if $(D_f)^2 = (D_{f^*})^2 = D^2$. These operators are no longer self-adjoint, $\overline{D_f} = D_{f^*}$, obey

$$\{D_f,\, D\} = 0\,, \quad \{D_{f^*},\, D\} = 0\,, \tag{12}$$

and commute with each other.

4. Symmetries due to families of unit roots

In what follows we look for special *families* of unit roots, $F = \{f^i \,|\, i = 1, 2, ..., N_F\} \subset \mathcal{R}_1(M_n)$, having supplementary properties which should guarantee simultaneously that: (I) the linear space $L_F = \{\xi \,|\, \xi = \xi_i f^i,\ \xi_i \in \mathbb{R}\}$ is isomorphic with a real Lie algebra, and (II) each element $\xi \in L_F - \{0\}$ is a root (of an arbitrary norm).

The first condition is accomplished only if the set $\{T(\xi) \,|\, \xi \in L_F\}$ includes a Lie group with N_F parameters. The corresponding Lie algebra of the generators has the real structure constants c_{ijk}:

$$[\left\langle f^i \right\rangle, \left\langle f^j \right\rangle] = c_{ijk} \langle f^k \rangle \,. \tag{13}$$

The condition (II) is accomplished only when $\langle \xi \rangle^2$ is equal up to a positive factor (i.e. the squared norm) with $-\mathbf{I}$. This requires to have

$$\left\{ \left\langle f^i \right\rangle, \left\langle f^j \right\rangle \right\} = -2\kappa_{ij}\, \mathbf{I} \,. \tag{14}$$

where κ is a *positive definite* metric that can be brought in canonical form $\kappa_{ij} = \delta_{ij}$ through a suitable choice of the unit roots. We have [8, 9]:

Theorem 2 *If the set $F = \{f^i \,|\, i = 1, 2, ..., N_F\} \in \mathcal{R}_1(M_n)$ is a family of unit roots then the matrices $\mathbf{I}$ and $\left\langle f^i \right\rangle$, $i = 1, 2, ..., N_F$, form the basis of a matrix representation of a finite-dimensional associative algebra over $\mathbb{R}$.*

Proof. If F is a family of unit roots in the sense of above definition then f^i must satisfy eqs. (13) and (14) with canonical metric. Hereby it results that the set of the real linear combinations $\xi_0 \mathbf{I} + \xi_i \left\langle f^i \right\rangle$ forms an associative algebra. This algebra is closed with respect to the matrix multiplication that can be calculated by adding the commutator and anticommutator. Moreover this algebra is a division one. There exists the zero element (with $\xi_0 = 0$, $\xi_i = 0$), the unit element is $\mathbf{I}$ and each element different from zero has the inverse $(\xi_0 \mathbf{I} + \xi_i \left\langle f^i \right\rangle)^{-1} = (\xi_0 \mathbf{I} - \xi_i \left\langle f^i \right\rangle)/({\xi_0}^2 + \xi_i \xi_i)$. Obviously, this real algebra is finite possessing a basis of dimension $N_F + 1$ where $\left\langle f^i \right\rangle$ play the role of complex units. □

This theorem severely restricts the existence of the families of unit roots. Indeed, according to the Frobenius theorem there are only two finite real algebras able to give suitable representations in spaces of roots, namely the algebra $\mathbb{C}$ of complex numbers and the *quaternion* algebra, $\mathbb{H}$. In the first case we have *isolated* unit roots f and representations of the $\mathbb{C}$ algebra generated by the matrices $\mathbf{I}$ and $\langle f \rangle$ (which play the role of $i \in \mathbb{C}$).

Here we focus on the second possibility leading to families of unit roots with $N_F = 3$ that constitute matrix representations of the quaternion units.

Theorem 3 *The unique type of family of unit roots with $N_F > 1$ having the properties (I) and (II) are the triplets $F = \{f^1, f^2, f^3\} \subset \mathcal{R}_1(M_n)$ which satisfy*

$$\left\langle f^i \right\rangle \left\langle f^j \right\rangle = -\delta_{ij}\, \mathbf{I} + \varepsilon_{ijk} \langle f^k \rangle \,, \quad i, j, k... = 1, 2, 3 \,. \tag{15}$$

Proof. Taking into account that ε_{ijk} is the anti-symmetric tensor with $\varepsilon_{123} = 1$ we recognize that eqs. (15) are the well-known multiplication rules of the quaternion units or similar algebraic structures (e.g. the Pauli matrices). Consequently, the matrices $\langle f^i \rangle$ and $\mathbf{I}$ generate a matrix representation of $\mathbb{H}$. The Frobenius theorem forbids other choices. □

In the case of triplets involving only real-valued unit roots when the geometry is hyper-Kähler, each family of real unit roots (i.e., a hypercomplex structure) F has its own Lie algebra $L_F \sim su(2)$. These algebras can not be embedded in a larger one because of the restrictions imposed by the Frobenius theorem. An example of hyper-Kähler manifold is the Euclidean Taub-NUT space which is equipped with only one family of real unit roots [10, 11] (see Appendix A). The manifolds with pseudo-Euclidean metric with odd n_+ and n_- have only pairs of *adjoint* triplets, F and F^*, the last one being formed by the adjoints of the unit roots of F. The spaces L_F and L_{F^*} are isomorphic between themselves (as real vector spaces) and all the results concerning the symmetries generated by F^* can be taken over from those of F using complex conjugation. Moreover, we must specify that the set $L_F \bigcup L_{F^*}$ is no more a linear space since the linear operations among the elements of L_F and L_{F^*} are not allowed. A typical example is the Minkowski spacetime which has a pair of conjugated triplets of complex-valued unit roots [6] (presented in Appendix B). Both these examples of manifolds possessing triplets with the properties (15) are of dimension four. As we know, the results indicate that similar properties could have all the flat manifolds of dimension $n = 4k$, $k = 1, 2, 3, ...$ where we expect to find many such triplets [12].

5. Gravitational anomalies

For the classical motions, a S-K tensor $K_{\mu\nu}$ generate a quadratic constant of motion as in eq. (3). In the case of the geodesic motion of classical scalar particles, the fact that $K_{\mu\nu}$ is a S-K tensor satisfying (1), assures the conservation of (3).

Passing from the classical motion to the hidden symmetries of a quantized system, the corresponding quantum operator analog of the quadratic function (3) is [13]:

$$\mathcal{K} = D_\mu K^{\mu\nu} D_\nu \,, \tag{16}$$

where D_μ is the covariant differential operator on the manifold with the metric $g_{\mu\nu}$. Working out the commutator of (16) with the scalar Laplacian

$$\mathcal{H} = D_\mu D^\mu$$

an explicit calculation gives [14]:

$$\begin{aligned}[]
[D_\mu D^\mu, \mathcal{K}] &= 2K^{(\mu\nu;\lambda)} D_\mu D_\nu D_\lambda + 3K^{(\mu\nu;\lambda)}{}_{;\lambda} D_\mu D_\nu \\
&+\{-\frac{4}{3} K_\lambda{}^{[\mu} R^{\nu]\lambda} + \frac{1}{2} g_{\lambda\sigma} (K^{(\lambda\sigma;\mu);\nu} - K^{(\lambda\sigma;\nu);\mu}) + K^{(\lambda\mu;\nu)}{}_{;\lambda}\}_{;\nu} D_\mu \,.
\end{aligned}$$

Note the very last terms is missing in the corresponding equation in [13].

Concerning the hidden symmetry of the quantized system, the above commutator does not vanishes on the strength of (1). If we take $K_{\mu\nu}$ to be a S-K tensor we are left with:

$$[\mathcal{H}, \mathcal{K}] = -\frac{4}{3}\{K_{\lambda}{}^{[\mu}R^{\nu]\lambda}\}_{;\nu}D_{\mu} \tag{17}$$

which means that in general the quantum operator $\mathcal{K}$ does not define a genuine quantum mechanical symmetry [15]. On a generic curved spacetime there appears a *gravitational quantum anomaly* proportional to a contraction of the S-K tensor $K_{\mu\nu}$ with the Ricci tensor $R_{\mu\nu}$.

It is obvious that for a Ricci-flat manifold this quantum anomaly is absent. However, a more interesting situation is represented by the manifolds in which the S-K tensor $K_{\mu\nu}$ can be written as a product of K-Y tensors $f_{\mu\nu}$ [2].

The integrability condition for any solution of (2), written for K-Y tensors of valence $r = 2$, is

$$R_{\mu\nu[\sigma}{}^{\tau}f_{\rho]\tau} + R_{\sigma\rho[\mu}{}^{\tau}f_{\nu]\tau} = 0\,.$$

Now contracting this integrability condition on the Riemann tensor for any solution of (2) we get

$$f^{\rho}{}_{(\mu}R_{\nu)\rho} = 0. \tag{18}$$

Let us suppose that there exist a *square* of the S-K tensor $K_{\mu\nu}$ of the form of a K-Y tensor $f_{\mu\nu}$ as in eq. (4). In case this should happen, the S-K equation (1) is automatically satisfied and the integrability condition (18) becomes

$$K^{\rho}{}_{[\mu}R_{\nu]\rho} = 0.$$

This relation implies the vanishing of the commutator (17) for S-K tensors which admit a decomposition in terms of K-Y tensors.

Using the S-K tensor components of the Runge-Lenz vector (A.8) we can proceed to the evaluation of the quantum gravitational anomaly for the generalized Taub-NUT metrics [4]. A direct evaluation [5] shows that the commutator (17) does not vanish.

In conclusion the operators constructed from symmetric S-K tensors are in general a source of gravitational anomalies for scalar fields. However, when the S-K tensor is of the form (4), then the anomaly disappears owing to the existence of the K-Y tensors.

6. Index formulas and axial anomalies

Atiyah, Patodi and Singer [16] discovered an index formula for first-order differential operators on manifolds with boundary with a non-local boundary condition. Their index formula contains two terms, none of which is necessarily an integer, namely a bulk term (the integral of a density in the interior of the manifold) and a boundary term defined in terms of the spectrum of the boundary Dirac operator. Endless trouble is caused in this theory by the condition that the metric and the operator be of "product type" near the boundary.

In [5] we computed the index of the Dirac operator on annular domains and on disk, with the non-local APS boundary condition. For the generalized Taub-NUT metrics [4], we found that the index is a number-theoretic quantity which depends on the metrics. In particular, our formula shows that the index vanishes on balls of sufficient large radius, but can be non-zero for some values of the parameters c, d (see Appendix A) and of the radius.

Theorem 4 *If* $c > -\frac{\sqrt{15}d}{2}$ *then the extended Taub-NUT metric does not contribute to the axial anomaly on any annular domain (i.e., the index of the Dirac operator with APS boundary condition vanishes).*

Proof. The proof of this statement can be found in Ref. [5]. □

The result is natural since the index of an operator is unchanged under continuous deformations of that operator. In our case this would amount to a continuous change in the metric. The absence of axial anomalies is due to the fact there exists an underlying structure that does not depend on the metric. However for larger deformations of the metric there could appear discontinuities in the boundary conditions and therefore the index could present jumps. Our formula for the index involves a computable number-theoretic quantity depending on the parameters of the metric.

We also examined the Dirac operator on the complete Euclidean space with respect to this metric, acting in the Hilbert space of square-integrable spinors. We found that this operator is not Fredholm, hence even the existence of a finite index is not granted.

We mentioned in [5] some open problems in connection with unbounded domains. The paper [17] brings new results in this direction. First we showed that the Dirac operator on $\mathbb{R}^4$ with respect to the standard Taub-NUT metric does not have L^2 harmonic spinors. This follows rather easily from the Lichnerowicz formula, since the standard Taub-NUT metric has vanishing scalar curvature. In particular, the index vanishes.

In conclusion for the axial anomaly the role of K-Y tensors is not so obvious. The topological aspects are more important and the absence of K-Y tensors does not imply the appearance of anomalies.

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Appendix A
Example: Euclidean Taub-NUT space

Let us consider the Taub-NUT space and the chart with Cartesian coordinates $x^\mu (\mu, \nu = 1, 2, 3, 4)$ having the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -f(r)(d\vec{x})^2 - \frac{g(r)}{16m^2}(dx^4 + A_i dx^i)^2 \,, \tag{A.1}$$

where $\vec{x}$ denotes the three-vector $\vec{x} = (r, \theta, \varphi)$, $(d\vec{x})^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ and $\vec{A}$ is the gauge field of a monopole $\mathrm{div}\vec{A} = 0\,,\quad \vec{B} = \mathrm{rot}\,\vec{A} = 4m\frac{\vec{x}}{r^3}$. The real number m is the parameter of the theory which enter in the form of the functions

$$f(r) = g^{-1}(r) = \frac{4m + r}{r} \tag{A.2}$$

and the so called NUT singularity is absent if x^4 is periodic with period $16\pi m$. Sometimes it is convenient to make the coordinate transformation $4m(\chi + \varphi) = -x^4$ with $0 \leq \chi < 4\pi$.

In the Taub-NUT geometry there are four K vectors [18, 12] and to them correspond the conservation of angular momentum and so called "relative electric charge":

$$\vec{j} = \vec{r} \times \vec{p} + q\,\frac{\vec{r}}{r}, \qquad q = g(r)(\dot{\theta} + \cos\theta\dot{\varphi}) \tag{A.3}$$

where $\vec{p} = f(r)\dot{\vec{r}}$ is the mechanical momentum.

On the other hand in the Taub-NUT geometry there are known to exist four K-Y tensors of valence 2. The first three are covariantly constant

$$\begin{aligned} & f_i = 8m(d\chi + \cos\theta d\varphi) \wedge dx_i - \epsilon_{ijk}(1 + \frac{4m}{r})\, dx_j \wedge dx_k, \\ & D_\mu f^\nu_{i\lambda} = 0\ , \quad i, j, k = 1, 2, 3. \end{aligned} \tag{A.4}$$

The f^i define three anticommuting complex structures of the Taub-NUT manifold, their components realizing the quaternion algebra

$$f^i f^j + f^j f^i = -2\,\delta_{ij}, \qquad f^i f^j - f^j f^i = -2\,\varepsilon_{ijk} f^k.$$

The existence of these K-Y tensors is linked to the hyper-Kähler geometry of the manifold and shows directly the relation between the geometry and the $N = 4$ supersymmetric extension of the theory [19, 20].

The fourth K-Y tensor is

$$f_Y = 8m(d\chi + \cos\theta d\varphi) \wedge dr + 4r(r + 2m)(1 + \frac{r}{4m}) \sin\theta d\theta \wedge d\varphi \tag{A.5}$$

having a non-vanishing covariant derivative $f_{Y\,r\theta;\varphi} = 2(1 + \frac{r}{4m})r\sin\theta$.
In Taub-NUT space there is a conserved vector analogous to the Runge-Lenz vector of the Kepler-type problem [18, 21, 22]

$$\vec{K} = \frac{1}{2}\,\vec{K}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = \vec{p}\times\vec{j} + \left(\frac{q^2}{4m} - 4mE\right)\frac{\vec{r}}{r} \tag{A.6}$$

where the conserved energy is $E = \frac{1}{2}\,g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$. The components $K_{i\mu\nu}$ involved with the Runge-Lenz vector (A.6) are S-K tensors and they can be expressed as symmetrized products of the K-Y tensors f_i, f_Y and K vectors of the space [23].

The *generalized* Taub-NUT manifolds [4] are defined by the line element (A.1) with the functions

$$f(r) = \frac{a+br}{r}\,, \qquad\qquad g(r) = \frac{ar+br^2}{1+cr+dr^2}\,. \tag{A.7}$$

depending on the arbitrary real constants a, b, c and d. If one takes the constants $c = \frac{2b}{a}\,, d = \frac{b^2}{a^2}$ the generalized Taub-NUT metric becomes the original Euclidean Taub-NUT metric (A.2) up to a constant factor.

The remarkable result of Iwai and Katayama [4] is that the generalized Taub-NUT space (A.7) admits a hidden symmetry represented by a conserved vector, quadratic in 4-velocities, analogous to the Runge-Lenz vector of the following form

$$\vec{K} = \vec{p}\times\vec{J} + \kappa\frac{\vec{x}}{r}\,. \tag{A.8}$$

The constant κ involved in the Runge-Lenz vector (A.8) is $\kappa = -a\,E + \frac{1}{2}\,c\,q^2$ where the conserved energy E is

$$E = \frac{\vec{p}^{\;2}}{2f(r)} + \frac{q^2}{2g(r)}\,.$$

The components $K_i = k_i^{\mu\nu}p_{\mu}p_{\nu}$ of the vector $\vec{K}$ (A.8) involve three S-K tensors $k_i^{\mu\nu}, \quad i = 1,2,3$ satisfying (1).

Appendix B
Example: Minkowski spacetime

In the Minkowski spacetime, $M_{(1,3)}$, with metric $\eta = (1,-1,-1,-1)$ we take the gauge $e^{\mu}_{\nu} = \hat{e}^{\mu}_{\nu} = \delta^{\mu}_{\nu}$, $(\mu,\nu,... = 0,1,2,3)$. We use the chiral representation of the Dirac matrices where the standard Dirac operator $D_s = i\gamma^{\mu}\partial_{\mu}$ reads

$$D_s = \left(\begin{array}{cc} 0 & i(\partial_t + \vec{\sigma}\cdot\vec{\partial}) \\ i(\partial_t - \vec{\sigma}\cdot\vec{\partial}) & 0 \end{array}\right) = \left(\begin{array}{cc} 0 & D^{(+)} \\ D^{(-)} & 0 \end{array}\right). \tag{B.1}$$

There are two families of three roots [7, 8, 9]. The unit roots of the first triplet, F, have the non-vanishing complex components [7]

$$f^{(1)}_{23} = 1\,, \qquad f^{(2)}_{31} = 1\,, \qquad f^{(3)}_{12} = 1\,, \tag{B.2}$$

$$f^{(1)}_{01} = i\,, \qquad f^{(2)}_{02} = i\,, \qquad f^{(3)}_{03} = i\,, \tag{B.3}$$

and, consequently, F is not a hypercomplex structure and the Minkowski spacetime is not a hyper-Kähler manifold. The corresponding spin-like operators

$$\Sigma^{(i)} = \frac{i}{4}\, f^{(i)}_{\mu\nu}\gamma^{\mu}\gamma^{\nu} = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix} \tag{B.4}$$

give Dirac-type operators. The second triplet is F^* for which all the spinor quantities are just the Dirac conjugated of those of F [8, 9].

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